Three Essays in Matching Mechanism Design

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Abstract

I consider the problem of allocating indivisible objects among agents according to their preferences when transfers are absent.

In Chapter 1, I study the tradeoff between fairness and efficiency in the class of strategy-proof allocation mechanisms. The main finding is that for strategy-proof mechanisms the following efficiency and fairness criteria are mutually incompatible: (1) Ex-post efficiency and envy-freeness, (2) ordinal efficiency and weak envy-freeness and (3) ordinal efficiency and equal division lower bound.

In Chapter 2, the focus is on two representations of an allocation when randomization is used: as a probabilistic assignment and as a lottery over deterministic assignments. To help facilitate the design of practical lottery mechanisms, we provide new tools for obtaining stochastic improvements in lotteries. As applications, we propose lottery mechanisms that improve upon the widely-used random serial dictatorship mechanism, and a lottery representation of its competitor, the probabilistic serial mechanism.

In Chapter 3, I propose a new mechanism to assign students to primary schools: the Adaptive Acceptance rule (AA). AA collects von Neumann-Morgenstern utilities of students over schools and implements the assignment using an iterative procedure similar to the prevalent Immediate Acceptance rule (IA). AA enjoys a strong combination of incentive and efficiency properties compared to IA and its rival, the Deferred Acceptance rule (DA). In case of strict priorities, AA implements the student-optimal stable matching in dominant strategies, which dominates each equilibrium outcome of IA. In case of no priorities, AA is ex-post efficient while some equilibrium outcomes of IA are not; also, AA causes loss of ex-ante efficiency less often than DA. If, in addition, students have common ordinal preferences, AA is approximately strategy-proof and ex-ante dominates DA.

Keywords:

probabilistic assignment, randomization, Random Serial Dictatorship, strategy-proofness, ex-post efficiency, envy-freeness, school choice, Immediate Acceptance, Deferred Acceptance

Zusammenfassung

In diese Dissertation, betrachte ich das Problem der Aufteilung der unteilbaren Objekte unter Agenten, ihren Vorlieben entsprechend, und die Transfers fehlen.

In Kapitel 1 studiere ich den Kompromiss zwischen Fairness und Effizienz in der Klasse der strategy-proof Aufteilungsmechanismen. Das wichtigste Ergebnis ist, dass für die strategy-proof Mechanismen folgende Effizienz- und Fairness-Kriterien nicht miteinander vereinbar sind: (1) Ex-post-Effizienz und Neidfreiheit, (2) Ordnung-Effizienz und schwache Neidfreiheit und (3) Ordnung-Effizienz und gleiche-Teilung-untere-Grenze.

In Kapitel 2 ist der Fokus auf zwei Darstellungen einer Zuteilung: als probabilistische Zuordnung und als Lotterie über deterministische Zuordnungen. Um die Gestaltung der praktischen Lotterie-Mechanismen zu erleichtern schlagen wir neue Werkzeuge für den Erhalt der stochastischen Verbesserungen bei Lotterien vor. Als Anwendungen schlagen wir Lotterie Mechanismen, die die weit verbreiteten Random serial dictatorship Mechanismus verbessern, und eine Lotterie-Darstellung seiner Konkurrent, die Probabilistic serial Mechanismus, vor.

In Kapitel 3 schlage ich einen neuen Mechanismus vor, der Schüler an Grundschulen zuweist: Adaptive Acceptance (AA). AA sammelt von Neumann-Morgenstern Präferenzen von Studenten über Schulen und implementiert die Zuordnung unter Verwendung eines iterativen Verfahrens, das ähnlich der vorherrschenden Immediate Acceptance (IA) ist. AA verfügt über eine starke Kombination von Anreize und Effizienzeigenschaften im Vergleich zu IA und sein Rivale, Deferred Acceptance (DA). Bei strengen Prioritäten setzt AA die Schüler-optimale-stabile-Paarung in dominanter Strategien um, die jedes Gleichgewichtsergebnis des IA dominiert. Ohne Prioritäten, ist AA Ex-post-effizient, während einige Gleichgewichtsergebnisse der IA nicht sind. Auch verursacht AA der Verlust der Ex-ante-Effizienz weniger häufig als DA. Wenn zusätzlich die Schüler gleiche ordinale Vorlieben haben, ist AA approximately strategy-proof und Ex-ante-dominiert DA.

Schlagwörter:

probabilistische Zuordnung, Randomisierung, Random Serial Dictatorship, strategy-proofness, Ex-post-Effizienz, Neidfreiheit, freie Schulwahl, Immediate Acceptance, Deferred Acceptance

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Introduction

The three essays constituting this dissertation study economic situations in which the usage of transfers such as money is not feasible. The importance of these situations, called matching problems, became evident in recent years as the number of real-life applications multiplied. At the same time, the infeasibility of transfers made the standard economic solutions such as markets and auctions unfit and thus gave rise to a new family of solutions: the matching mechanisms. These mechanisms are based solely on the reported preferences of agents. And the main challenge is for each situation to find a suitable mechanism that is efficient, fair, and robust to manipulation. The design of such mechanisms is the focus of my dissertation.

What differentiates the three chapters in this dissertation is the undertaken approach. The first chapter takes a constructive approach and proposes a new algorithmic solution for one of the large-scale matching problems – the school choice problem. The second chapter takes an instrumental approach and develops tools to construct welfare-enhanced matching mechanisms. The third chapter takes an axiomatic approach and shows which combinations of properties of a mechanism are feasible and which are not.

What all three chapters have in common is that the matching mechanisms considered here involve randomization. Randomization is a typical way to restore fairness (in case the preferences of some agents are conflicting) without weakening the robustness to manipulation. However, randomization has its costs: adding randomness can result in welfare losses. The first chapter proposes a mechanism that – compared to the traditionally used mechanisms – involves less randomization and thus restores a certain amount of the welfare loss. The second chapter studies the interplay between two representations of mechanisms: one that is useful for ex-ante analysis – before the uncertainty is resolved – and the other that is useful for ex-post analysis. The third chapter studies the tradeoff of feasible efficiency and fairness properties at various levels of randomization. I briefly summarize the main findings of the three chapters below.

In the first chapter, the focus is on the school choice problem, which is among the most studied and relevant problems in the matching literature. In this problem, the question is how to assign students to primary schools. Traditionally, this allocation is done based on the ordinal preferences of students over schools and the coarse priorities of schools over students. The prevalent mechanism – the Immediate Acceptance mechanism (IA), also known as the Boston mechanism – is very manipulable and often induces inefficient assignments. Its main competitor – the Deferred Acceptance mechanism (DA) – makes truthful reporting dominant, but can have unambiguous efficiency losses from an ex-ante perspective because of added randomness through the random tie-breaking in the coarse priorities of schools.

I propose a new school choice mechanism: the Adaptive Acceptance rule (AA). Different from IA and DA, AA collects von Neumann-Morgenstern preferences of students over schools. Based on this additional information, AA implements the assignment using an iterative procedure that combines the algorithms of IA and DA. As a result, AA enjoys a strong combination of incentive and efficiency properties compared to IA. At the same time, compared to DA, AA breaks fewer ties in priorities and thus adds less randomness. Specifically, the three mechanisms are comparable in two stylized cases: when schools have strict priorities or when they have no predetermined priorities. In case of strict priorities, AA implements the student-optimal stable matching in dominant strategies, which dominates each equilibrium outcome of IA. In case of no priorities, AA is ex-post efficient while some equilibrium outcomes of IA are not; also, AA causes loss of ex-ante efficiency less often than DA. If, in addition, students have common ordinal preferences, AA is non-manipulable in large economies and ex-ante dominates DA. For the case of arbitrary priorities, I run simulations, which support the main findings. I find that the simplified version of AA restores up to approximately half of the welfare loss associated with DA.

In the second chapter, the focus is on two representations of the expected assignment: as a matrix of assignment probabilities (which is useful for ex-ante analysis in terms of efficiency and fairness), and as a lottery over deterministic assignments (which is useful for ex-post analysis and is commonly used for indivisible goods allocation in real life). To help facilitate the design of practical lottery mechanisms, we provide new tools for obtaining stochastic improvements in lotteries. As applications, we propose lottery mechanisms that improve upon the widely used Random Serial Dictatorship mechanism and a lottery representation of its competitor, the Probabilistic Serial mechanism. The tools we provide here can be useful in developing welfare-enhanced new lottery mechanisms for practical applications such as school choice.

In the third chapter, I consider the standard object allocation problem: how to allocate N indivisible objects among N agents. This problem is the foundation for the school choice problem, among other problems. The focus here is the tradeoff between fairness and efficiency in the class of non-manipulable mechanisms. There are several degrees of efficiency that a mechanism can satisfy. For example, a mechanism can be ex-post efficient – induce a lottery over Pareto-efficient deterministic assignments, and ordinally efficient – induce a probabilistic assignment that is not first-order stochastically dominated. These notions of efficiency are logically ordered: the latter implies the former. There are also several types of fairness properties that a mechanism can satisfy. For example, a mechanism can satisfy equal division lower bound – such that each agent prefers the assignment to the equal division. A mechanism can be weak envy-free – such that no agent's assignment is first-order stochastically dominated by some other agent's assignment. A mechanism can also be envy-free – such that each agent's assignment first-order stochastically dominates each other agent's assignment, and this notion implies the previous two. The main finding is that for non-manipulable mechanisms there is a tradeoff between fairness and efficiency. Specifically, the following properties are mutually incompatible: (1) ex-post efficiency and envy-freeness, (2) ordinal efficiency and weak envy-freeness, and (3) ordinal efficiency and equal division lower bound. Result 1 is the first impossibility result for this setting that uses ex-post efficiency; results 2 and 3 are more practical than similar results in the literature.

Chapter 1

School Choice With Advice: The Adaptive Acceptance Rule

This chapter is based on Nesterov (2015).

1.1 Introduction

Each year millions of students around the world enter primary schools. In recent decades an increasing number of cities granted students the right to choose their primary school by adopting so-called centralized school choice programs. The cornerstone of each of these programs is the allocation rule — a systematic procedure that collects the preferences of students over schools, as well as the schools' priorities over students[1] and derives the resulting allocation. These allocation rules are the main focus of this paper. Since the formal introduction of the *school choice problem* by Abdulkadiroğlu and Sönmez (2003) there has been an extensive debate as to which rules should be used in real-life, leading to the replacement of rules in several cities in the US and around the world.[2]

[1]Priorities are not the preferences of schools, but only the rights of students to be accepted by some schools prior to other students.

[2]See recent surveys on school choice problem by Abdulkadiroğlu (2013) and Pathak (2011). Extensive description of the school choice and other matching programs in Europe can be found at www.matching-in-practice.eu.

In this paper we propose a novel solution to the school choice problem — the Adaptive Acceptance rule (AA). The main feature of AA is that it explicitly accounts for the *intensities* of students' preferences. Imagine two students preferring the same school over other schools, but the first student is almost indifferent, while the second student prefers even a tiny chance of getting into her favorite school over a guaranteed seat in any other school. All established rules can only treat these two students equally as they collect only *ordinal* preferences. In contrast, AA collects the students' von Neumann-Morgenstern (vNM) preferences, which reflect the value of *lotteries* over schools. As a consequence, AA enjoys a strong combination of fairness in Plato's sense,[3] Pareto efficiency, as well as incentives for truthful preference reporting — compared to other existing rules.[4]

AA works as follows: first students report their vNM preferences over schools and then the allocation is found using an iterative procedure with multiple rounds. (Hereafter we refer to students' choices and actions, even though these are pseudo-students operated by AA algorithm as the "true" students only submit their preferences and the rest is done by AA acting on their behalf.) At the beginning of the first round, on behalf of each student, the mechanism applies to the student's top-ranked school. If the number of applicants at every school does not exceed the school's capacity, then each school assignes seats among its applicants and the algorithm terminates. However, if at some school the number of the applicants exceeds the school's capacity, then each school *conditionally* assigns its seats among applicants up to its capacity according to its priority. As a result, some stu-

[3]"When equality is given to unequal things, the resultant will be unequal" (Plato, *Laws*). A similar formula is due to Aristotle: "Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences" (Nicomachean Ethics). Similar ideas have been expressed by a range of thinkers including Thomas Jefferson, Felix Frankfurter and Anton Menger.

[4]The only other school choice rule based on vNM utilities is the Competitive Equilibrium with Equal Incomes rule (CEEI) introduced by Hylland and Zeckhauser (1979). This rule has superior efficiency and fairness properties, as well as strong incentives for truth-telling in large markets. The generalized for the school choice context version of CEEI is due to He, Miralles, Pycia and Yan (2015). However, the comparison with CEEI is complicated by the fact that it is a non-single-valued correspondence as there might be multiple equilibria and finding any of them is non-trivial. That is why here we restrict ourselves only to comparison with IA and DA.

dent is conditionally assigned a seat with probability one, another student is conditionally assigned with probability zero. Besides, as some schools might have *weak* priorities and the ties are broken at random,[5] some students are conditionally assigned a seat with probability between zero and one.

At this point each student knows only the expected utility of his current application, as well as the expected utility of an *alternative* application. That is, if he unilaterally withdraws his current application and applies to some other school (possibly with a higher assignment probability). If the student prefers to change his application, we say that he prefers to *switch*. If he is the only student who prefers to switch, we switch him to his most preferred application. If there are several students who prefer to switch, then we select one of these students using a specific choice rule and switch him or her.

We repeat the last step again and again — evaluate the expected utilities from current and alternative applications for each student, pick a student who prefers to switch and switch him or her — until no student prefers to switch.[6] Finally, each school assigns its seats among its applicants up to its capacity according to its priority, the ties in priorities are broken at random. Assigned students leave the procedure, while others proceed to the next round. In general, each round begins with the students applying to their favorite remaining schools, then students switch until nobody prefers to switch, the seats are assigned among applicants, and so forth until each student is assigned.[7]

Let us illustrate how the AA procedure works using the following example, in which all students have the same priority at all schools. Moreover, students have common ordinal preferences over schools: they all prefer school A over school B, and school B over school C, while the intensities of their preferences differ.

Example 1.1. Let there be three students Andy, Ivan, and Jenny and three schools A, B, and C with one seat each. Students have the following vNM

[5] Throughout the paper we assume i.i.d. uniform random tie-breaking.

[6] This series of switchings always converges due to Lemma 1.1.

[7] The precise definition of the AA rule consists of the AA game and the AA algorithm and is given in section 1.3.

utilities over schools:

	Andy	Ivan	Jenny
A	8	6	6
B	2	4	4
C	0	0	0

,

AA	Andy	Ivan	Jenny
A	0,5	0,25	0,25
B	0	0,5	0,5
C	0,5	0,25	0,25

.

In the beginning of the first round of the AA procedure each student applies to school A and receives it with probability $\frac{1}{3}$. Conditional on that nobody prefers to switch, each student expects to get a seat at school B and school C in the later rounds with probability $\frac{1}{3}$, and thus the expected payoff of the default application is $\frac{10}{3}$ for each student. However, two students —Ivan and Jenny— prefer to switch to school B as it brings them a higher expected payoff of 4. Let us (randomly) select Ivan and change his application to school B. As nobody else wants to switch, we reach the end of the round and break ties at random. Ivan is assigned to school B, while either Andy or Jenny is assigned to school A (each with probability $\frac{1}{2}$). The unassigned student gets a seat at school C in the next round.

Does AA incentivize truth-telling? In the example we saw that both Ivan and Jenny preferred to switch from A to B but only if they did it alone. Also, each of them prefers to switch, rather than to stay, as switching to B brings the expected payoff of 4 while staying at A brings only 3. Clearly, either Ivan or Jenny could manipulate the report by top-ranking B and applying to it directly. Therefore AA is manipulable. However, as the economy grows large, the incentives to manipulate AA in this case (given a certain choice rule) vanish.[8]

How efficient is AA compared to other rules? To make this comparison, let us consider the two most studied and applied school choice rules: the Immediate Acceptance rule (IA) and the Deferred Acceptance rule.[9] These

[8]See section 1.2 for precise definition of approximate strategy-proofness and Proposition 1.4 for the result.

[9]These rules remain our benchmarks for the entire paper, the precise definitions of the two rules are given in section 2. IA is often called the Boston mechanism as it was used in Boston when first discovered and described by Abdulkadiroğlu and Sönmez (2003).

three rules can be compared in two specific cases: if schools have strict priorities and if schools have no priorities.

AA is strongly related to both of these rules and even, if slightly modified, coincides with them. If ties in priorities are publicly broken ex-ante, and the assignment is made only at the end of the last round, then AA coincides with DA. If students apply in k-th round of AA to their k-th ranked school and are not allowed to switch, then AA coincides with IA.[10] These differences in procedures have strong implications for the students' strategic behavior and for the induced assignment. Let us illustrate these two rules and compare them to AA using the same example.

DA is non-manipulable and we can assume that students report truthfully — then each of them receives equal assignment probability $\frac{1}{3}$ for each school. In contrast, as IA is manipulable even in large markets, we usually focus on its equilibrium outcomes. The unique symmetric equilibrium of the game associated with IA is in mixed strategies: students Ivan and Jenny skip school A and apply directly to B with probability 0.61. The (probabilistic) assignments of DA, IA, and the matrix of expected payoffs $\mathbb{E}(u)$ for the three rules are as follows:

DA	Andy	Ivan	Jenny
A	0,33	0,33	0,33
B	0,33	0,33	0,33
C	0,33	0,33	0,33

,

IA	Andy	Ivan	Jenny
A	0,66	0,17	0,17
B	0,05	0,47	0,47
C	0,29	0,35	0,35

,

$\mathbb{E}(u)$	Andy	Ivan	Jenny
DA	3,3	3,3	3,3
IA	5,4	2,9	2,9
AA	4	3,5	3,5

.

As a result, only Andy prefers the IA outcome over the DA outcome, while Ivan and Jenny are worse off since they often end up applying to B

[10] If students apply to the best remaining school but still are not allowed to switch, then AA coincides with an improved version of IA studied by Dur (2013), and Harless (2015). Mennle and Seuken (2014) compare this improved version of IA with the standard IA in case all students have the same priority at each school.

together. In contrast to IA, AA helps Ivan and Jenny to coordinate correctly so that precisely one of them skips A and applies to B directly.

We show that this observation holds in general: if students have common ordinal preferences and schools are indifferent between students, all students ex-ante prefer the outcome of AA over the outcome of DA.[11]

In a more general case, when schools are still indifferent between students, but students have arbitrary ordinal preferences over schools, we show two results regarding the efficiency of AA. First, the deterministic outcome of AA is Pareto efficient, while some of the IA equilibria are not. Second, AA causes less efficiency loss ex-ante compared to DA. Specifically, for each problem where the DA assignment is not stochastically dominated, the AA assignment is not stochastically dominated either, while the reverse is not true. We further analyze the case in which schools are indifferent between students in section 1.4.

We also study the opposite extreme of the schools' priorities — if each school has strict priority ordering over all students. In this case we show that, similar to DA, AA induces the student optimal stable matching (SOSM) in dominant strategies. Ergin and Sönmez (2006) show that in the strict priority case SOSM dominates *each* equilibrium outcome of IA, and therefore AA and DA dominate IA. These results are presented in section 1.5.

In order to compare the efficiency properties of AA and DA in case of general priorities, we simulate a random environment (as in Erdil and Ergin, 2008). For the ease of computing, we use a simplified AA rule in which the continuation option is calculated assuming that nobody switches in future rounds, and these expectations are updated after every switch. For each simulation we also compute the welfare-maximizing assignment and compare it to the outcomes of DA and the simplified AA. The simulations show that the simplified AA welfare dominates DA and that it restores up to approxi-

[11]Abdulkadiroğlu, Che and Yasuda (2011) show the same result for all symmetric equilibria of IA. Here ex-ante means the point at which each student observes her preferences but before she observes the preferences of other students. Troyan (2012) extends this result for the case of different priorities by taking a more ex-ante perspective: given common ordinal preferences, before each student gets to know her utilities and priorities, she prefers any symmetric equilibrium outcome of IA over DA.

mately half of the welfare loss associated with DA. The specification of the simulation model and the results are presented in section 1.6.

Section 1.7 discusses implementation of AA and concludes.

1.2 Framework

We have two types of agents: students and schools. Students are strategic players with strict preferences over schools but they also might face uncertain outcomes — lotteries over deterministic assignments to schools.[12] In order to represent the students' preferences over lotteries, we endow each student with vNM preferences over schools. In addition, each student has an outside option — to remain unassigned (e.g., to find the school outside the school choice program or to go to a private school). From each student's vNM equivalence class we select his utility profile such that the outside option brings him a zero utility.

The distinctive feature of the school choice problem is that some students are entitled to be assigned to a certain school *prior* to some other students. In other words students have different *priorities* at different schools or, less formally, schools have different priorities over students. We do not interpret these priorities as preferences of schools over students since schools are taken as mere objects and in what follows we define efficiency and incentive properties only from the perspective of the students. Yet, these priorities might be respected in certain ways depending on the problem and the solution concept that we use. We now introduce the model formally.

1.2.1 Model

Let I be a set of students and S be a set of schools including the outside option s_0 (e.g., going to a private school).

Each student $i \in I$ receives a vNM utility u_{is} if he is assigned a seat in school $s \in S/\{s_0\}$ and $u_{is_0} \equiv 0$. He draws a **utility vector** $u_i \equiv (u_{is})_{s \in S}$ from

[12] Students might face uncertain outcomes because of random tie-breaking in priorities of schools and incomplete information about other participating students.

a finite set of utility vectors $U \equiv \{(u_1, ..., u_{|S|}) \mid \forall s, s' \in S\backslash\{s_0\}\ u_s \neq u_{s'}\}$ according to some distribution $f(\cdot)$.[13] Let $u \equiv (u_i)_{i\in I}$ denote the **utility profile**.

Each school $s \in S$ has a capacity of $\kappa_s \in \mathbb{N}$ seats; capacity of the outside option is $\kappa_{s_0} \equiv |I|$. Let $\kappa \equiv (\kappa_s)_{s\in S}$ denote the **capacity profile**. Also, s has a **priority** $\succsim_s$ — a weak linear order over I. Let $\succsim_S \equiv (\succsim_s)_{s\in S}$ denote the **priority profile**.

We call the tuple $(I, S, \kappa, u, \succsim_S)$ a school choice problem, or simply a **problem**. In what follows I, S and κ are fixed, and we vary only the utility profile and the priority profile.

Given I, S and κ, a matrix of non-negative numbers $P \equiv \{P_i\}_{i\in I} \equiv \{P_{is}\}_{i\in I, s\in S}$ such that for each $i \in I$ $\sum_s P_{is} = 1$ and for each school $s \in S$ $\sum_i P_{is} \leq \kappa_s$ is a **(probabilistic) assignment**, whith P_{is} being interpreted as the probability with which i receives a seat in s. If all elements in P are either zero or one, we call P a **matching**. Let $\mathcal{P}$ denote the set of all assignments.

Efficiency Given $P, P' \in \mathcal{P}$, student i **prefers** P **to** P' if P gives i a higher expected utility than P': $\sum_s u_{is}P_{is} > \sum_s u_{is}P'_{is}$; i is **indifferent between** P **and** P' if $\sum_s u_{is}P_{is} = \sum_s u_{is}P'_{is}$. We say that P **dominates** P' if each student either prefers P to P' or is indifferent between P and P'. Also, P **strictly dominates** P' if P dominates P' and at least one student prefers P to P'. Finally, P is **ex-ante efficient** if it is not strictly dominated by any other assignment in $\mathcal{P}$; P is **ex-post efficient** if it can be written as a lottery over ex-ante efficient matchings.

Similarly, P **sd-dominates** P' if for each $i \in I$, given i's ordinal preferences derived from u_i, P_i first-order stochastically dominates P'_i. We say that P is **sd-efficient** if it is not sd-dominated by any other $P' \in \mathcal{P}$.

A **rule** is a mapping φ that for each problem gives an assignment $(u, \succsim_S) \overset{\varphi}{\mapsto} P \in \mathcal{P}$, denoted $\varphi(u, \succsim_S) \equiv \{\varphi_i(u, \succsim_S)\}_{i\in I} \equiv P$. A rule φ is **ex-ante efficient (ex-post efficient, sd-efficient)** if it always gives an ex-ante efficient (ex-post efficient, sd-efficient) assignment.

[13] $|S|$ denotes the cardinality of set S.

Incentive compatibility In the preference revelation game associated with some rule φ, each student i's strategy is a utility u_i and his payoff is his expected utility from his assignment $\varphi_i(u_i, u_{-i})$.[14] We are interested in rules such that no student ever benefits from misreporting his preferences. Rule φ is **strategy-proof** if for each problem, each $i \in I$ has a higher expected payoff if he reports his true u_i than if he reports any other $u'_i \in U$: $\sum_s u_{is}\varphi_{is}((u_i, u_{-i}), \succsim_S) \geq \sum_s u_{is}\varphi_{is}((u'_i, u_{-i}), \succsim_S)$.

We are also interested in rules in which incentives to misreport preferences vanish in large problems. Let us replicate each student $i \in I$ and each seat in each school $s \in S$ for m times.[15] A rule φ is **approximately strategy-proof** if for each problem $(u, \succsim_S)$, each positive $\varepsilon > 0$, there exists $\bar{m} \in \mathbb{N}$ such that for each $m \geq \bar{m}$, each student $i \in I$, and each $u'_i \in U$: $u_i\varphi_i((u'_i, u_{-i}), \succsim_S) - u_i\varphi_i((u_i, u_{-i}), \succsim_S) < \varepsilon$.

Next we define the two benchmark rules—the Deferred Acceptance rule and the Immediate Acceptance rule—and briefly discuss their properties.

1.2.2 Deferred Acceptance Rule

The Deferred Acceptance rule (DA) is based on the algorithm of the same name defined below. This algorithm works only with strict preferences and priorities.

Definition. Given a utility profile u and a strict priority profile $\succ_S$, the DA algorithm works as follows:

Round 1: Each student i applies to the school with the highest utility u_{is}; each school s, among all its applicants, tentatively accepts students up to its capacity κ_s according to its priority $\succ_s$; every other student is rejected.

....

Round r: Each student rejected at the previous round applies to his next most preferred school; each school s, among all its applicants *and* pre-

[14] The subscript $(-i)$ denotes all students in I except student i: u_{-i} denotes the report of all students except student i.

[15] Replicating student i for m times means adding m students of the same utility and priority as student i; replicating a seat in school s for m times means increasing the capacity in s to $(m + 1)\kappa_s$.

viously accepted students, tentatively accepts students up to its capacity κ_s according to its priority $\succ_s$; every other student is rejected.

....

The algorithm terminates when no student is rejected; each student tentatively accepted at some school is assigned a seat at this school. The other students receive the outside option. □

If priorities are strict, DA induces the unique student optimal stable matching (SOSM)[16] (Gale and Shapley, 1962). However, if priorities are not strict, the ties are randomly broken ex-ante.[17] If the ties are broken using a single lottery for all schools, then we call the rule induced by the DA algorithm DA with single tie-breaking (DA-S). If the ties are broken using multiple lotteries that are not consistent for different schools — we call the rule DA with multiple tie-breaking (DA-M). If priorities are not strict, there is no unique undominated stable matching. Moreover, DA-M may induce a dominated stable matching as the tie-breaking creates artificial constraints respected by the DA algorithm.[18]

1.2.3 Immediate Acceptance Rule

The Immediate Acceptance rule (IA) is based on the algorithm of the same name defined below. Unlike the DA algorithm, the IA algorithm assigns the seats in demanded schools as quickly as possible.[19]

Definition. Given a utility profile u and a priority profile $\succsim_S$, IA algorithm works as follows.

[16]A matching is called stable if no student can find a better seat than the one she gets that is either free, or is assigned to a student of lower priority.

[17]Hereinafter we write only random tie-breaking as ties will always be broken using uniform distribution.

[18]One of the undominted stable matchings can still be found in polynomial time using the "stable improvement cycles algorithm" due to Erdil and Ergin, 2008. However, this algorithm distorts the students' incentives to report their preferences truthfully, while DA is strategy-proof (Dubins and Freedman, 1981; Roth, 1982). In fact, no strategy-proof rule dominates DA (Kesten and Kurino, 2015).

[19]Here the distinction between the single tie-breaking and the multiple tie-breaking is more subtle compared to the case of the DA algorithm since the ties are broken at the moment of assignment. We stick to the multiple tie-breaking in order to make the comparison with AA algorithm simpler.

Round 1: Each student i applies to the school with the highest utility u_{is}; each school s assigns seats among applicants up to its capacity $\kappa_s^1 \equiv \kappa_s$ according to its priority $\succsim_s$ (ties are broken at random); remaining capacity is decreased by the number of assigned students and denoted as κ_s^2; each unassigned student is rejected.

....

Round r: Each unassigned student applies to his next school with the highest utility (his r-ranked school); each school s assigns seats among applicants up to its remaining capacity κ_s^r according to its priority $\succsim_s$ (ties are broken at random); remaining capacity is decreased by the number of assigned students and denoted as κ_s^{r+1}; each unassigned student is rejected.

....

The algorithm terminates when all students are assigned. □

The major problem with IA is that it is manipulable even in large markets as some students have strong incentives to skip their top-ranked schools if they have a low assignment probability at these schools; each of these students will apply to some other schools with a higher assignment probability prior to other students and distort the assignment. The game induced by IA algorithm is very complex and usually has multiple equilibria. Yet, the efficiency comparison between the DA outcome and the IA equilibrium outcomes is ambiguous; we further discuss the efficiency comparison in sections 4 and 5 for special versions of the school choice problem.

We now proceed to the main result of the paper and introduce the Adaptive Acceptance rule.

1.3 Adaptive Acceptance Rule

For each problem the AA rule finds an assignment. For clarity of exposition we split the AA rule in two parts: the AA game and the AA algorithm. In each part we act on students' behalf, but we continue using the term "students" for simplicity. The real students only submit their vNM utilities and then receive the assignment. Similarly, the term "game" is used by the analogy with economic game in order to simplify the exposition of the mechanism.

The *AA game* is a Bayesian game with multiple rounds. In each round each student applies to only one school and all participating students do so simultaneously. Then each school assigns its seats among its applicants up to its capacity according to its priority. The ties are broken randomly according to i.i.d. uniform distribution. Each school's remaining capacity is decreased by the number of assigned seats. The unassigned students proceed to the next round.

In each round of the AA game a specific equilibrium application in pure strategies is determined using the *AA algorithm*. The AA algorithm has multiple steps. In the first step each student applies to her most preferred school. Then we evaluate the expected payoff of each student from her application and identify students who want to apply to a different school — *to switch*. Among all students who want to switch we select one student and switch this student to the school with the highest expected payoff, keeping other applications fixed. This ends the first step. In the second step we again identify students who want to switch, select one of them, switch and then proceed to the next step. We do the same again and again until no student wants to switch.

The intuition behind these two foundations is the same as in DA and IA. The AA game is similar to the game induced by the IA algorithm: all demanded seats are assigned according to the priorities, the ties are broken in each round, and this assignment is final. The AA algorithm, on the other hand, is similar to the DA algorithm as it prescribes the order of applications: each student first applies to her most preferred school and then some students are "rejected" (they receive a zero or a low assignment probability) and therefore apply to a different school with a higher expected payoff.

Let us consider the process of switching in the AA algorithm in more detail. The first ingredient in this process is the individual decision of a student to switch in each particular step. Each student i decides to switch based on two factors: the current assignment probability at each available school and the expected payoff in the next round of the AA game. This expected payoff we call *the continuation option*, which i expects to get if he is not assigned in the current round and continues to play in the next round. The continuation option matters since if student i has optimistic expectations about the future rounds, he is willing to accept higher risks at the current round. And vice versa: if the continuation option is not that good, i would prefer to get a safer option in the current round, i.e., to apply to a school with higher assignment probability.[20] The correct value of the continuation option is determined by the AA algorithm, we discuss this in more detail when we define the algorithm.

The second ingredient in the process of switchings is the order of switchings, which is determined using a specific *choice rule.* Given the set of students who want to switch in a particular step, the choice rule selects a single student and switches her to the school with the highest expected payoff.[21] The choice rule can involve randomization (e.g., in order to break ties between otherwise symmetric students) but only ex-ante—before the first step of the first round—so that the only remaining uncertainty in the AA game comes from the random tie-breaking in priorities and everything else is determined by the AA algorithm.[22]

For each problem $(u, \succsim_S)$ we define the AA game, the AA algorithm and the AA rule in consecutive order after we introduce necessary notation.

[20]The decision to switch is myopic as each student does not take into account the subsequent switchings of other students. We choose this design so as to simplify the algorithm. An alternative design would account for the subsequent switchings and give a corresponding advice.

[21]If more than one school has the highest expected payoff, we select one of them using the same choice rule.

[22]The AA algorithm uses the multiple tie-breaking — for the following two reasons. First, we want the strategy of each student i to be simpler: the decision whether to switch to school s depends only on the current number of applicants at s that have a same or higher priority. In contrast, the single tie-breaking rule would make such decisions dependent on i's beliefs about his position in the original lottery relative to the positions of his potential competitors — applicants at s with the same priority. In the first round these beliefs are straightforward but then, after each tie-breaking, each students updates his beliefs about his and the other students' positions in the lottery, which unnecessarily complicates the decision process. Secondly, in order to maximize welfare in each round of AA, we want the students with relatively higher intensities for some schools to apply to these schools. Multiple tie-breaking accomplishes this task, while single tie-breaking might bring certain distortions in students' decisions due to their beliefs.

For each round r, **the set of unassigned students** is I^r, **the remaining capacity** at school s is κ_s^r, and **the set of remaining schools** S^r is the set of schools with positive remaining capacity $S^r = \{s \in S|\kappa_s^r > 0\}$. The vector of capacities is denoted as $\kappa^r \equiv \{\kappa_s^r\}_{s\in S^r}$.

In each round r of the AA game, the **application** a^r is a function $I^r \overset{a^r}{\mapsto} S^r$ such that each student $i \in I^r$ applies to some school $s \in S^r$ denoted as $s \equiv a^r(i)$. The set of students applying to school s as $a^r(s)$. The set of all admissible applications in round r is A^r.

Given a^r and $\succsim_{S^r}$, the set of participating students in the next round $I^{r+1} \subset I^r$ is determined by tie-breaking. Let us denote the set of all admissible sets I^{r+1} by $\{I\}^{r+1}$: I^{r+1} is drawn from $\{I\}^{r+1}$ using a uniform distribution (since the ties at each school are broken using i.i.d. uniform distribution). Let us denote the set of all admissible sets I^{r+1} such that $i \in I^{r+1}$ as $\{I\}_i^{r+1}$.

Given a^r, κ^r and $\succsim_S$, the **assignment probability** of student i at school s is denoted as $\pi_{is} \equiv \pi_{is}(a^r, \kappa^r, \succsim_S)$, where if $s = a(i)$ then π_{is} is the assignment probability of the current application and if $s \neq a(i)$ then π_{is} is the assignment probability of an alternative application.

The next notions we define recursively starting from the last round. In the last round $\bar{r}$, given $a^{\bar{r}}$, each student $i \in I^{\bar{r}}$ receives a payoff $(v_{ia^{\bar{r}}(i)})^{\bar{r}} \equiv \pi_{ia^{\bar{r}}(i)}u_{ia^{\bar{r}}(i)}$. In round $\bar{r}-1$ student i's continuation option $x_i^{\bar{r}-1}$ is his expected payoff in the next round $\bar{r}$, in which he among other students $I^{\bar{r}} \in \{I\}_i^{\bar{r}}$ competes for remaining schools $S^{\bar{r}}$: $x_i^{\bar{r}-1} \equiv \frac{1}{|\{I\}_i^{\bar{r}}|}\sum_{I^{\bar{r}}\in\{I\}_i^{\bar{r}}}(v_{ia^{\bar{r}}(i)})^{\bar{r}}$ (where the equal weights $\frac{1}{|\{I\}_i^{\bar{r}}|}$ are due to the i.i.d. random tie-breaking at each school). In general, in each round r, student i's **continuation option** x_i^r is his expected payoff in round $r+1$, in which a set of students $I^{r+1} \in \{I\}_i^{r+1}$ competes for the set of remaining schools S^{r+1}: $x_i^r \equiv \frac{1}{|\{I\}_i^{r+1}|}\sum_{I^{r+1}\in\{I\}_i^{r+1}}(v_{ia^{r+1}(i)})^{r+1}$. We denote the **continuation profile** of all students in I^r in round r as $x^r \equiv \{x_i^r\}_{i\in I^r}$; $X^r = \{x^r\}$ denotes the set of all admissible continuation options. In each round r, student i's **expected payoff** is $(v_{ia^r(i)})^r \equiv \pi_{ia^r(i)}u_{ia^r(i)} + (1-\pi_{ia^r(i)})x_i^r$.

We now formally define the AA game.

AA GAME In each round $r = 1, 2, ...$, given I^r, S^r, κ^r, each $i \in I^r$ chooses an application $a^r(i)$ from S^r that gives him the expected payoff $(v_{ia^r(i)})^r$. If i receives a seat, he leaves the game, otherwise he proceeds to the next round.

The AA game differs from the IA game in that each student chooses one application in each round, while in the IA the order of applications is determined by the preference list.

Next we define the necessary notation for the AA algorithm.

For each step t of round r, for each $i \in I^r$ his current application in step t of round r is $a_t^r(i)$. Student i's **current payoff** is the expected value of his application to $s \in S^r$ plus the expected value of the next round: $(v_{is})_t^r \equiv \pi_{is} u_{is} + (1 - \pi_{is}) x^r$. Also, student $i \in I^r$ **prefers to switch** from his current school $a_t^r(i)$ if there exists at least one school $s' \in S^r$ such that application to s' promises a higher current payoff: $(v_{is'})_t^r > (v_{ia(i)})_t^r$. The set of all students who prefer to switch we denote as D_t^r. Finally, the **choice rule** C is a deterministic function that for each D_t^r selects a student $i \in D_t^r$ and a school $s \in S^r \backslash \{a_t^r(i)\}$ which has the highest current payoff: for each school $s' \in S$ $(v_{is})_t^r \geq (v_{is'})_t^r$ (if several schools have the same highest current payoff, C selects one of them deterministically). In what follows we fix the choice rule C.

In order to find the perfect Bayesian Nash equilibrium in each round of the AA game we have to run the switching procedure in the AA algorithm multiple times. We first start with "optimistic" continuation options for each student: each student expects that only the seats that are requested in the first step will be assigned. We check, whether this is an equilibrium. If it is, the algorithm terminates. If it is not an equilibrium, and some student prefers to switch, we pick a less optimistic continuation option.

The order in which we pick the continuation profiles is defined using the algorithm itself: we start with $\bar{x}$ such that the number of switches is zero and thus the number of assigned seats is the lowest. (For example, if $\bar{x}$ is such that each student $i \in I^r$ expects to get in round $r+1$ the payoff that is as good as his most preferred school $s \in S^r$, then i never prefers to switch.) Then we pick a continuation option x such that only one more seat is assigned (thus x has to be lower than $\bar{x}$ at least for one student). If for some group

$\{x\} \in X^r$ the number of assigned seats is the same, we arbitrary order this group. Let us denote the resulting application of the AA algorithm given continuation option x as $AA(x)$ and the corresponding number of assigned seats as $|AA(x)|$. Then for each $x, x' \in X^r$ we say that $x \geq_X x'$ if the number of assigned seats in the AA algorithm given x is weakly lower than given x': $|AA(x)| \leq |AA(x')|$ and, for all $x, x' \in X$ such that $|AA(x)| = |AA(x')|$ the order is defined arbitrary.

We can now define the AA algorithm.

AA ALGORITHM For each $r = 1, 2, ...$ the application a^* is determined as follows.

Round r**:** Given I^r, S^r, κ^r, pick the first x^r from X^r according to $\geq_X$.

Step 1. Each student applies to his most preferred school; the initial distribution of applications a_1^r is determined. Given a_1^r and x^r, we determine the set D_1^r of students who prefer to switch. Using choice rule C, we select a student-school pair $(i, s) = C(D_1^r)$ and i switches from his current school $a(i)$ to s.

....

Step t. Given a_t^r and x^r, determine D_t^r, select the student-school pair using choice rule $C(D_t^r)$ and implement the switch.

....

Step $\bar{t}$: When no student prefers to switch, $D_{\bar{t}}^r = \emptyset$, if $x(a_{\bar{t}}^r) \leq_X x^r$ then terminate and $a^* \equiv a_{\bar{t}}^r$, otherwise pick the next smallest x^r from X^r according to $\geq_X$.

Finally, the AA rule is defined as the equilibrium application in the AA game, which is determined by the AA algorithm.

AA RULE Formulate the AA game and for each round $r = 1, 2, ...$, find an equilibrium application a^* using the AA algorithm; assign seats in each school $s \in S^r$ among applicants $a^*(s)$ according to $\succsim_s$ (break ties at random).

We illustrate the AA rule using the following example.

Example 1.2. Let there be four students and four schools with one seat each, all schools are indifferent between students and students have the following vNM utilities:

	1	2	3	4
s_1 :	90	90	90	90
s_2 :	8	8	6	2
s_3 :	2	2	4	8
s_4 :	0	0	0	0

.

We first pick x such that no student prefers to switch in the first round and $a^* : a^*(1) = a^*(2) = a^*(3) = a^*(4) = s_1$. Given that, we apply the AA algorithm in the second round, find the true x^1 and check whether $AA(x^1) = a^*$.

Let us first find x_1^1: the expected payoff of student 1 after the first round if one of the other three students received s_1. There are three cases:

(i) if student 2 receives s_1, then in the second round students 1 and 3 apply to s_2, student 4 applies to s_3 and no student prefers to switch; the expected payoff of student 1 equals 4;

(ii) if student 3 receives s_1, then in the second round students 1 and 2 apply to s_2, student 4 applies to s_3 and no student prefers to switch; the expected payoff of student 1 equals 4;

(iii) if student 4 receives s_1, then in the second round students 1,2 and 3 apply to s_2 and only student 3 prefers to switch to s_3; the expected payoff of student 1 equals 4.

Summing up these expected payoffs with equal weights $\frac{1}{3}$ we get $x_1^1 = 4$. By analogy, we find $x_2^1 = 4$, $x_3^1 = \frac{10}{3}$ and $x_4^1 = 8$. Given these continuation option, no student i prefers to switch to any other school $s \neq s_1$ in the first round as he would receive a lower expected payoff than if he applies to s_1: $u_{is} < (\frac{1}{4}u_{is_1} + \frac{3}{4}X_i^1)$.[23]

[23]In fact, this inequality holds for any positive x_i^1 and we did not need to find the precise values of the continuation options.

Conditional on who receives s_1 in the first round, we find the outcome of the second round and the resulting assignment is:

AA	1	2	3	4
s_1 :	1/4	1/4	1/4	1/4
s_2 :	3/8	3/8	1/4	0
s_3 :	0	0	1/4	3/4
s_4 :	3/8	3/8	1/4	0

.

This example also illustrates the difference between the AA outcome and any of the IA equilibrium outcomes. In the unique equilibrium of the IA game, the optimal strategy of student 3 is to report s_2 as the second-best school. The resulting IA assignment coincides with the DA assignment. Compared to AA, this strategy of student 3 harms both him and students 1 and 2 and does not benefit student 4.

IA, DA	1	2	3	4
s_1 :	1/4	1/4	1/4	1/4
s_2 :	1/3	1/3	1/3	0
s_3 :	1/12	1/12	1/12	3/4
s_4 :	1/3	1/3	1/3	0

,

$\mathbb{E}(u)$	1	2	3	4
AA	25,5	25,5	25	28,5
IA	25,3(3)	25,3(3)	24,8(3)	28,5
DA	25,3(3)	25,3(3)	24,8(3)	28,5

.

Let us consider a more complex example in which some students prefer to switch in the first round.

Example 1.3. Let there be four students and four schools with one seat each, all schools are indifferent between students and students have the following vNM utilities, where ε is small and positive:

	1	2	3	4
s_1 :	3	3	4	4
s_2 :	2	2	1	1
s_3 :	$1-\varepsilon$	$1-\varepsilon$	$1-\varepsilon$	$1-\varepsilon$
s_4 :	0	0	0	0

.

We first pick x such that no student prefers to switch in the first round and $a^* : a^*(1) = a^*(2) = a^*(3) = a^*(4) = s_1$. Given that, we apply the AA algorithm in the second round, find the true x^1 and check whether $AA(x^1) = a^*$.

Let us first find x_1^1: the expected payoff of student 1 after the first round if one of the other three students received s_1. There are three cases:

(i) if student 2 receives s_1, then in the second round students 1 applies to s_2, students 3 and 4 prefer to switch to s_3; the expected payoff of student 1 equals 1;

(ii) if student 3 receives s_1, then in the second round students 1 and 2 apply to s_2, student 4 prefers to switch to s_3; the expected payoff of student 1 equals 1;

(iii) if student 4 receives s_1, then in the second round students 1 and 2 apply to s_2, student 3 prefers to switch to s_3; the expected payoff of student 1 equals 1.

Summing up these expected payoffs with equal weights $\frac{1}{3}$ we get $x_1^1 = 1$. But this is not compatible with a^* as student 1 prefers to switch to s_2.

Next we pick x such that only one student switches. W.l.o.g., let it be student 1: $AA(x) = ((1, s_2), (2, s_1), (3, s_1), (4, s_1))$. If no other student prefers to switch then the continuation option of other students are: $x_2^1 = x_3^1 = x_4^1 = \frac{1-\varepsilon}{2}$. Indeed, given this x^1 no other student prefers to switch and $AA(x^1)$ is an equilibrium application. Assuming that student 2 could have deviated instead of student 1, the resulting assignment is

AA	1	2	3	4
s_1 :	1/6	1/6	1/3	1/3
s_2 :	1/2	1/2	0	0
s_3 :	1/6	1/6	1/3	1/3
s_4 :	1/6	1/6	1/3	1/3

.

This assignment dominates the DA assignment, in which each student gets any seat with probability 1/4. However, the assignment is not ex-ante efficient since there is a Pareto improving exchange of probability shares for schools s_1, s_2 and s_4.

Next we show that the AA rule is a well-defined mapping.

Theorem 1.1. *The AA rule is well-defined.*

The proof of the Theorem relies on two results presented as lemmas. The first lemma shows that for each continuation option x the algorithm always converges to an equilibrium application given x.

Lemma 1.1. *For any problem, for each continuation profile $x \in X$, the switching process in the AA algorithm converges to allocation $a \equiv AA(x)$ such that a is an equilibrium given x.*

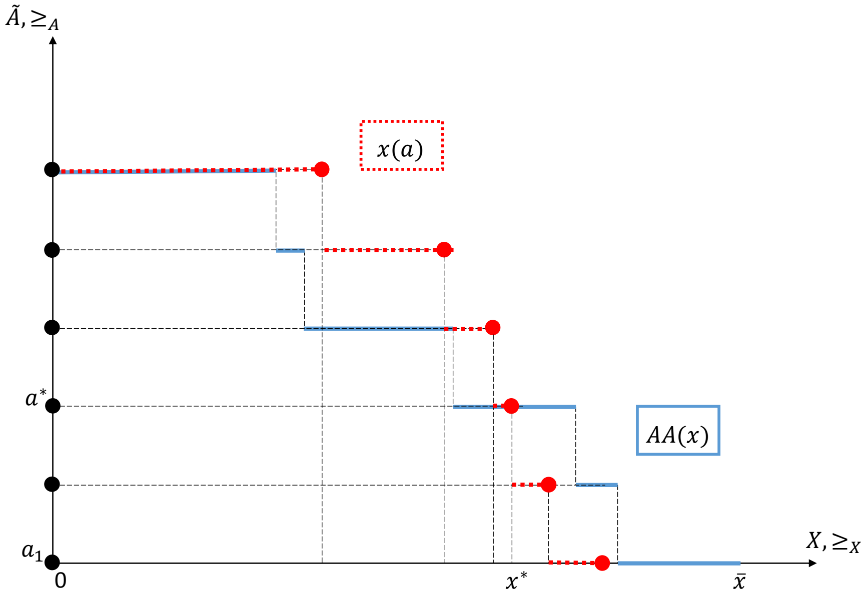
Proof. See the Appendix. □

Lemma 1.1 shows that $AA(x)$ is a well-defined mapping from the set of continuation options X to the set of applications A. Denote the image of this mapping as $\widetilde{A}$. Based on the ordering $\geq_X$ we define a corresponding ordering $\geq_A$ on $\widetilde{A}$. For each $x, x' \in X$ such that $x \geq_X x'$ their images in $\widetilde{A}$ have the same relation $AA(x) \geq_A AA(x')$.

We assume that the mapping $x(a)$ is such that as the number of demanded seats $|a|$ increases (and thus the number of remaining seats decreases), the continuation option weakly decreases. We say that $x(a)$ is **regular** if for each $a, a' \in \widetilde{A}$ such that $a \geq_A a'$ their images have the same relation $x(a) \geq_X x(a')$.

Since we define relation $\geq_A$ indirectly via $AA(\cdot)$, and we are also agnostic about $AA(\cdot)$ as it is determined by choice funcion C, it is difficult to precisely interprete the regularity condition beyond what it says above. Roughly

Figure 1.3.1: Graphical representation of mappings AA(x) and x(a).



Notes: The horizontal axis represents set X according to $\geq_X$, the vertical axis represents set $\tilde{A}$ according to $\geq_A$. The solid step-function represents mapping $AA(x)$, the bold points connected by dotted line represent mapping $x(a)$.

speaking, the regularity condition requires the continuation option to depend on the set of remaining seats, rather than on the set of remaining students.

Next we show that if the mapping $x(a)$ is regular, then the AA algorithm always finds a fixed point x^*.

Lemma 1.2. *For any problem, if the mapping $x(a)$ is regular, then the AA algorithm finds x^* such that $x(AA(x^*)) = x^*$.*

Proof. To see that, consider Figure 1.3. On the horizontal axis we plot set X according to $\geq_X$, on the vertical axis we plot set $\widetilde{A}$ according to $\geq_A$. By definition of $\geq_A$, the graph representing $AA(\cdot)$ is a decreasing step function. By regularity condition, the graph representing $x(a)$ is also a decreasing step function.

Let us compare the extreme points of the two graphs. Let $\bar{x}$ be such that no student wants to deviate (e.g., assume that each student expects to get his second best school in the next round). Then each student applies to his best school $AA(\bar{x}) = a_1$, the application in the first step of the AA algorithm. The corresponding continuation option $x(a_1)$ lies to the left: $\bar{x} \geq_X x(a_1)$ since each student can expect at most his second best school in the next round. In the other extreme each student expects to get the outside option in the next round. Then the continuation option $x(AA(0))$ lies to the right: $x(AA(0)) \geq_X 0$. Finally, since the domain of $x(\cdot)$ is the image of $AA(\cdot)$, the graphs have at least one intersection and the AA algorithm picks the highest of them. □

Regularity is a sufficient condition: if $x(a)$ is not regular, the AA algorithm might still find an equilibrium x^*. Otherwise the AA algorithm terminates at the first intersection of the two graphs.

In the following sections we compare the properties of the three rules: DA, IA, and AA. We first do it for the two extreme cases that have been studied in the literature: the extremely coarse priority case (as in Abdulkadiroğlu, Che and Yasuda, 2011, and Miralles, 2008) is considered in section 4; the strict priority case (as in Ergin and Sönmez, 2006) is considered in section 5. In section 6 we present simulation results for a general case (as in Erdil and Ergin, 2008).

1.4 No Priority Case

1.4.1 Arbitrary Ordinal Preferences

In this section we consider the school choice problem where each school is indifferent between all students.[24] Given this assumption, we compare AA and DA in terms of ex-post efficiency, sd-efficiency and ex-ante efficiency.

The case of extremely coarse priorities is of particular relevance as in reality schools often do not have predetermined priorities over students. One of the prominent examples is the within-district school choice program for

[24]This version of the school choice problem is equivalent to the object with multiple copies allocation problem.

middle schools in Beijing[25] Another relevant application is allocation of school seats in down town: as the majority of students comes from other districts, many seats will be allocated among students who have the same priority at each of these schools.

We begin our analysis with *ex-post efficiency*. If schools are indifferent between all students, DA-S coincides with the random priority rule, which is ex-post efficient. In contrast, DA-M is not ex-post efficient as priorities of schools create artificial constraints respected by DA algorithm.[26] Some equilibria of IA are also not ex-post efficient as shown by Example 1.6 in the Appendix.

Similar to DA with a single tie-breaking, AA always induces only efficient matchings:

Proposition 1.1. *In no priority case, AA is ex-post efficient.*

Proof. See the Appendix. □

Let us now take the ex-ante perspective and focus on *sd-efficiency*. The next example demonstrates a problem for which AA is sd-efficient, while DA is not.

Example 1.4. (Adopted from Bogomolnaia and Moulin, 2001). Let there be four students and four schools with one seat each. Schools are indifferent between students, and students have the following von Neumann-Morgenstern (vNM) utilities:

	1	2	3	4
s_1 :	7	7	2	2
s_2 :	2	2	7	7
s_3 :	1	1	0	0
s_4 :	0	0	1	1

.

[25] He (2012) studies the strategic behavior of students under IA using the data from one of the districts in Beijing.

[26] See Abdulkadiroğlu and Sönmez, 2003 for details.

The AA algorithm proceeds as follows: in the first round students $1, 2$ apply to their most preferred school s_1, students $3, 4$ apply to their most preferred school s_2 and no student wants to switch. The ties are broken, two students are assigned, and each of the two remaining students applies to the best of the remaining schools s_3, s_4.

The assignments induced by AA and DA are as follows:

AA	1	2	3	4
s_1 :	1/2	1/2	0	0
s_2 :	0	0	1/2	1/2
s_3 :	1/2	1/2	0	0
s_4 :	0	0	1/2	1/2

,

DA	1	2	3	4
s_1 :	5/12	5/12	1/12	1/12
s_2 :	1/12	1/12	5/12	5/12
s_3 :	5/12	5/12	1/12	1/12
s_4 :	1/12	1/12	5/12	5/12

.

Observe that the AA assignment first-order stochastically dominates the DA assignment. (The same holds for the unique equilibrium of IA.)

This example illustrates the case in which AA is ex-ante efficient (and thus sd-efficient) while DA is not. We show that AA and DA can be ordered in terms of efficiency in the following way: AA inefficiency for a certain problem always implies DA inefficiency for this problem, while the opposite is not true as demonstrated in Example 1.4. We summarize these results in the following Proposition.

Proposition 1.2. *In no priority case, if for some problem DA induces an sd-efficient (ex-ante efficient) assignment, then AA also induces an sd-efficient (ex-ante efficient) assignment.*

Proof. See the Appendix □

Proposition 1.2 provides a partial comparison between AA and DA in terms of efficiency: AA causes an efficiency loss in a strict subset of cases where DA causes such a loss. Yet, Proposition 1.2 does not tell us the magnitude of this effect. In the next subsection we impose an additional constraint on the preferences of students and show that AA ex-ante dominates DA. We also show that in this case AA is approximately strategy-proof given a specific choice rule.

1.4.2 Common Ordinal Preferences

Let us now consider a more restricted version of the school choice problem with extremely coarse priorities, in which students have common ordinal preferences over schools. This assumption is often relevant as real-life schools can be ranked similarly by all families because of the objective quality of schools. W.l.o.g., we also assume that the number of schools and the number of students are the same and that each school has just one seat.

In this special case, any symmetric equilibrium outcome of IA ex-ante dominates the DA outcome.[27] Ex-ante here means the moment at which students observe their own types, but do not yet know the types of other students. We show an analogous result for AA.

Proposition 1.3. *In no priority, common ordinal preference case, AA ex-ante dominates DA.*

Proof. See the Appendix □

We now discuss the incentive properties of AA in this special case. For this purpose, we first propose a specific choice rule C^* that simplifies the process of switchings as none of the students will ever switch upward her rank-ordered list.

W.l.o.g., consider the first round: each student first applies to s_1 and if any student wants to switch, then he wants to switch to s_2. If there are several students who want to switch from s_1 to s_2, consider the following choice rule. Pick student i_1 for whom $\lambda_{i_1}^{12} \equiv \frac{u_{i_1 s_2} - x_{i_1}}{u_{i_1 s_1} - x_{i_1}}$ is among the highest: $\lambda_{i_1}^{12} \geq \lambda_{i'}^{12}$ for any other student i'; and switch i_1 to s_2.

Now, if any other student wants to switch, it can only be another student at s_1 who wants to go either to s_2 or s_3. If there is at least one student who wants to switch to s_3, pick student i_2 for whom $\lambda_{i_2}^{13} = \frac{u_{i_2 s_3} - x_{i_2}}{u_{i_2 s_1} - x_{i_2}}$ is the highest: $\lambda_{i_2}^{13} \geq \lambda_{i_2}^{13}$ for any other student $i' \neq i_1$. If it turns out that $\lambda_{i_2}^{13} < \lambda_{i_1}^{13}$, then we make i_2 switch to s_2 and let i_1 switch to s_3. If the latter is not the case, then we let i_2 switch to s_3.

[27]For this special case the comparison between IA and DA has been made by Abdulkadiroğlu, Che and Yasuda (2011), as well as by Miralles (2008).

If there is no student who wants to switch to s_3, then there is one who wants to switch to s_2. Then we pick a student i_2 for whom $\lambda^{12}_{i_2} = \frac{u_{i_2 s_2} - x_{i_2}}{u_{i_2 s_1} - x_{i_2}}$ is among the highest: $\lambda^{12}_{i_2} \geq \lambda^{12}_{i'}$ for any other student $i' \neq i_1$. Let i_2 switch to s_2. If after that either i_2 or i_1 want to switch from s_2 to s_3, let the one with the higher λ^{23} switch and the other stay at s_2. We do the latter in order to reduce the "tension" at s_2 so that the only students who want to switch are among those applying to s_1.

We continue in the same way until no student wants to switch from s_1. In general, among students who want to switch, we select those who want to switch to the lowest school s and among these students we pick student i with the highest λ^{1s}_i. If some other student j at a school s' other than s_1 has a higher $\lambda^{1s}_j > \lambda^{1s}_i$, then we switch j to s and fill his vacant position using a series of switchings as it is done above. This defines the choice rule C^*.

Importantly, none of the students at an earlier step switched downwards will ever want to switch upwards at a later step. This is the characteristic property of C^*: each student i ever applying to s_2 (or another school different from s_1) has a higher λ^{12}_i than λ^{12}_j of any student j who switched from s_1 later than i. Therefore, i's expected gain from switching to s_1 is strictly lower than j's, but j decided to switch from s_1 and thus i cannot prefer to switch back to s_1. The same holds for all other pairs of schools due to the construction of C^*.

Therefore we have shown the following lemma.

Lemma 1.3. *In no priority, common ordinal preference case, given the choice rule C^*, all students in AA algorithm switch only downwards.*

Let us now consider the incentive properties of AA for this particular choice rule C^*. Recall from Example 1.1 that even in this special case AA never satisfies strategy-proofness. However, we can show that incentives to switch in AA with the choice rule C^* vanish as the market becomes large.

Proposition 1.4. *In no priority and common ordinal preference case, given the choice rule C^*, AA is approximately strategy-proof.*

Proof. See the Appendix. □

In the next section we consider the other extreme version of priority profile — if each school has a strict priority over all students.

1.5 Strict Priority Case

In this section we consider the school choice problem in which each school has a strict priority ordering over the set of students. We then compare AA, DA, and IA in terms of efficiency and incentives in this special case.

The assumption of strict priorities becomes relevant in settings when priorities of schools are fine: when students are prioritized based on their IQ scores or other standardized test results.

We first show that if priorities strict, AA induces the student optimal stable matching.

Proposition 1.5. *In strict priority case AA induces student optimal stable matching.*

Proof. Consider the AA procedure. In the beginning, as in DA, each student applies to her most preferred school. Since the priorities of schools are strict, the assignment probability of each student is either zero or one. Those who have probability one are preliminary accepted and will not switch in the moment. Each student i who has a zero probability will prefer to switch to the best school at which he has assignment probability one. This differs from the original Gale-Shapley student-proposing algorithm only in that student i does not necessarily apply to his second (third and so on) best school as this school can be occupied by another student j with a higher priority. However, if that school is occupied by some student j, it will remain occupied by the same student j or some other student j' with even higher priority than i has. Therefore, AA and DA algorithms are equivalent. □

Since DA is strategy-proof, we draw an immediate conclusion regarding the incentives for truthful reporting in AA:

Corollary 1.1. *In strict priority case AA is strategy-proof.*

The important feature of the strict priority case is that we can perfectly order the DA and AA outcomes and all of the IA equilibria in terms of efficiency. Consider the following motivating example.

Example 1.5. Let there be two students 1,2 and two schools s_1, s_2 with one seat each. Let the students' preferences and the schools' priorities be as follows:

1	2
s_1	s_2
s_2	s_1

,

s_1	s_2
2	1
1	2

.

The IA game has two equilibrium outcomes: $(1, s_1), (2, s_2)$ and $(1, s_2), (2, s_1)$. DA and AA, on the other hand, both induce the former dominant outcome: each student first applies to her favorite school, none of the students is rejected or prefers to switch and this assignment is final.

This observation for DA holds more generally: if preferences of students and priorities of schools are strict, the set of equilibria outcomes under IA coincides with the set of stable matchings.[28] This occurs due to that in each equilibrium of IA all students play a complex anti-coordination game. For each stable matching, consider the following strategy: each student reports just one school that she gets in this stable matching. This is an equilibrium strategy since no student can profitably change the report: applying to another school could benefit this student if and only if this creates a blocking pair, which would violate stability.

Depending on how well students solve this anti-coordination problem in the IA game, the equilibrium is more or less beneficial for students (and vice versa for schools if priorities are taken as preferences of schools). Since AA induces the student optimal stable matching in case priorities are strict, we get a clear efficiency ranking of AA and IA:

Corollary 1.2. *In strict priority case AA dominates all equilibria outcomes of IA.*

[28]This result is due to Ergin and Sönmez (2006); Pathak and Sönmez (2008) extend this result for the case where the population of students is mixed in terms of strategic sophistication.

Next we turn back to the general case in which priorities are intermediately coarse. As this case is of particular interest and complexity, we run a series of simulations to evaluate properties of AA compared to DA.

1.6 Simulations for the General Case

So far we compared AA to DA and IA for special problems in which the priorities are either extremely strict or extremely weak. In this section we consider a more general intermediate case of coarse priorities: schools are not completely indifferent between students and also do not have strict priorities. We restrict our comparison to AA and DA as these two rules are deterministic while IA is not and it is not easy to consistently predict which equilibrium will students coordinate on in the IA game.

For the ease of computing, we use a simplified version of the AA algorithm. Here, in each round of the AA game, each student has a continuation option determined by AA assuming that nobody switches in the next rounds. This option continuation option is updated after each step. For example, in the first step of the first round each student expects that only the currently demanded seats are assigned in this round. If some student switches to an empty seat, the continuation option for each other student (weakly) decreases as there are less seats to be assigned in the next round.

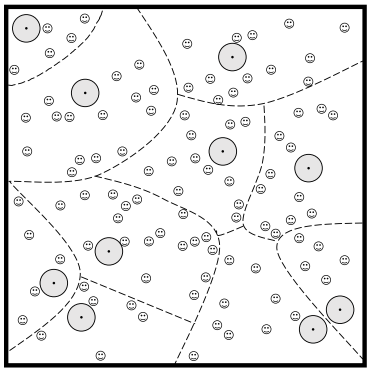
We design a random environment similar to the one used by Erdil and Ergin (2008). Being simple enough, the design accommodates a number of realistic features of the real-life school choice problems. One of these features is that the students' preferences over schools are correlated to some degree, which reflects the variation of schools in objective quality. The second feature deals with the intrinsic preferences over schools: the preferences depend on the distance to a school and on the individual taste different for each student. Finally, the priorities of schools are made consistent with the real-life situations: each student gets a higher priority only at the school she lives close to.

Imagine a unit square city with $N = 100$ students and $N_S = 10$ schools, each school s with capacity $\kappa_s = 10$ seats, as shown in the next figure.

All students and schools are located at random, according to i.i.d. uniform distribution.

Each student receives a priority 1 at the nearest school and a priority 0 at all other schools. The set of students is thus divided into 10 disjoint subsets, each of them having a higher priority at a particular school and an equally low priority at other schools.

The utility u_{is} of each student i for a school s consists of three ingredients. The first ingredient is the Euclidean distance between student i and school s, we denote this distance by $||s-i||$. It is only natural to assume that, ceteris paribus, students prefer schools that are located closer to them as smaller distances mean, for example, smaller transportation time and costs, higher security, higher average number of familiar neighbors in that school, and so forth.



Unit square city.
Smileys represent students, gray circles represent schools, dash lines partition city into school districts.

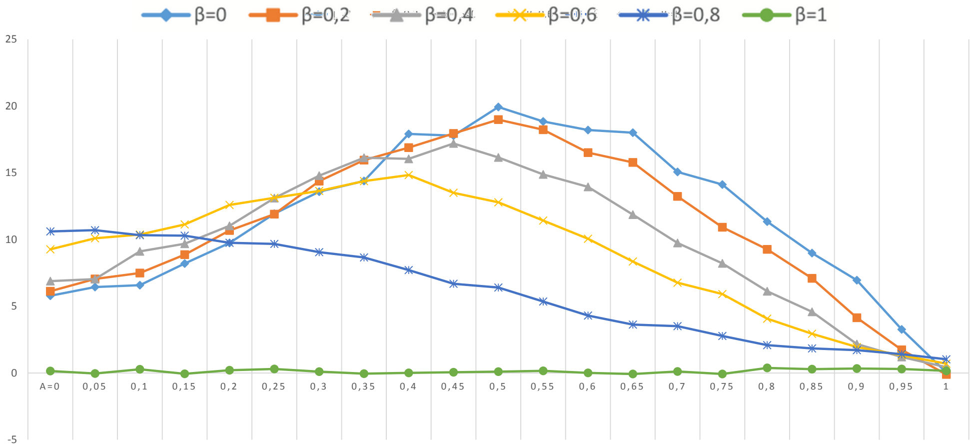
The second ingredient is the a common preference Z_{0s} for each school s, which is normally distributed with a a zero mean and a unit variance. This ingredient reflects the objective quality of school s compared to other schools. This quality is common information that students observe from publicly available sources such as school rankings, news or rumors.

Finally, the third ingredient Z_{is} is the individual preference of student i for school s, which is also normally distributed with a zero mean and a unit variance. The individual preference reflects either the intrinsic taste of student i for school s, or i's private information about the quality of school s.

These three ingredients are weighted using parameters α and β such that $0 \leq \alpha, \beta \leq 1$: α reflects the correlation of tastes and β reflects the transportation costs. The preference of student i for school s is then:

$$u_{is} = -\beta||s-i|| + (1-\beta)(\alpha Z_{0s} + (1-\alpha)Z_{is}).$$

Figure 1.6.1: Net number of students preferring AA over DA for different transportation costs β.



Notes: The horizontal axis corresponds to the correlation parameter α, the vertical axis corresponds to the number of students that prefer AA over DA minus the number of students that prefer DA over AA.

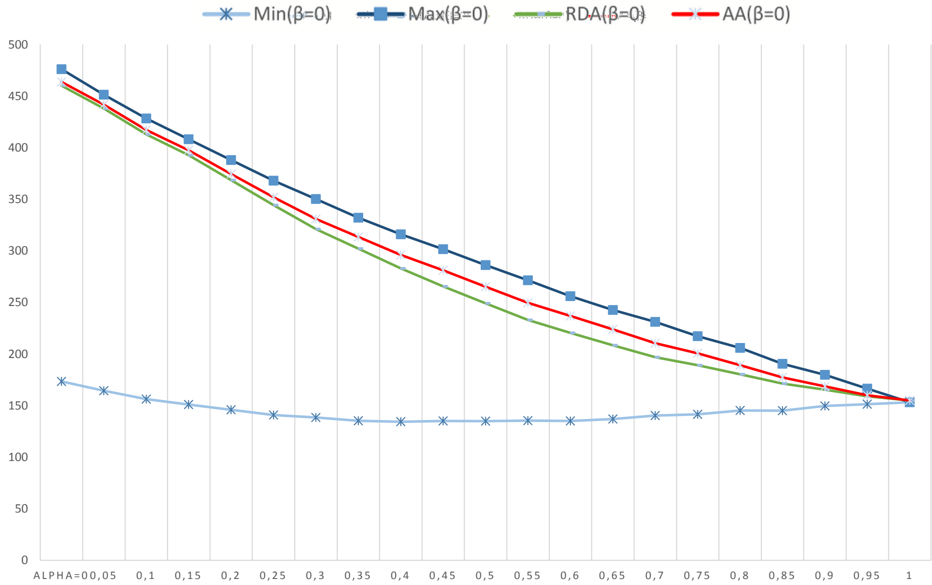
For each pair of the different preference parameters (α, β), we randomly locate students and schools and draw Z_{0s}, Z_{is} to simulate preferences of each student i for each school s. Student i gets a higher priority at the school that is located closest to him. Given the preferences and priorities and a random tie-breaking rule, we find the DA and AA assignments.[29] (For tractability, in AA we approximate each student's continuation option as the outcome of IA in future rounds.) We then repeat the simulation 200 times in order to get a consistent result.

We compare the performance of DA and AA for a given combination of the correlation parameter α and transportation costs parameter β.

First we observe that AA does not Pareto improve upon DA as some students prefer the DA outcome over the AA outcome. The students who are worse off under AA than under DA are those who received a higher priority for a decent, though not their most preferred, school. Under DA, this school becomes their reserve option which they can always take if they are unlucky at the better schools. In contrast, under AA, this reserve option is not available anymore since it might be taken in one of the first rounds.

[29]We use the single tie-breaking rule for DA and the multiple tie-breaking rule for AA.

Figure 1.6.2: Average welfare of AA and DA given $\beta = 0$.



Notes: The horizontal axis corresponds to the correlation parameter α, the vertical axis corresponds to welfare defined as the sum of all students' utilities. Lines from top to bottom: welfare maximizing assignment, AA, DA, welfare minimizing assignment.

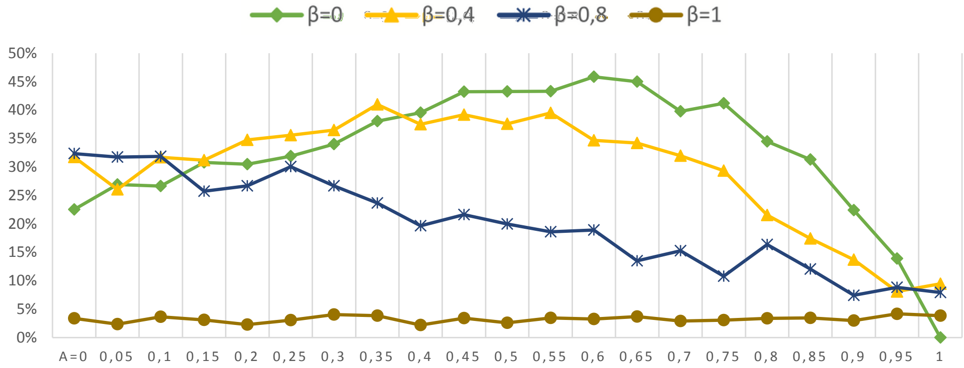
The number of students that prefer AA over DA is always larger than the number of students who prefer DA over AA. Figure 1.6.1 demonstrates the difference between these two, i.e., how many more students are better off under AA than under DA.

Next we compare the welfare under the two rules. To this end, we sum up all the individual utilities of students and find their averages for different parameters α and β. As there is no natural welfare benchmark, for our case we determine the average welfare of the welfare-minimizing assignment $\bar{W}_{min}$ and the average welfare of the welfare maximizing-assignment $\bar{W}_{max}$.

Figure 1.6.2 shows the average welfare of AA $\bar{W}_{AA}$ and the average welfare of DA $\bar{W}_{DA}$ compared to the extreme values $\bar{W}_{min}$ and $\bar{W}_{max}$ in case of zero transportation costs $\beta = 0$. We see that DA already performs quite well compared to the welfare-minimizing assignment, yet AA consistently improves upon DA.

Figure 1.6.3 shows how much of the relative welfare loss associated with DA is restored under AA: $(\bar{W}_{AA} - \bar{W}_{DA})/(\bar{W}_{max} - \bar{W}_{DA})$.

Figure 1.6.3: Welfare gain of AA compared to DA welfare loss.



Notes: The horizontal axis corresponds to the correlation parameter α, the vertical axis corresponds to the share of welfare loss associated with DA that is recovered by AA.

1.7 Discussion and Conclusion

In this paper we introduce a new school choice rule, the Adaptive Acceptance rule (AA), and compare it to the Deferred Acceptance rule (DA) and the Immediate Acceptance rule (IA).

The main feature of AA compared to DA and IA is the explicit use of cardinal preferences. Based on the reported preferences, the AA algorithm helps each student to find the best application strategy given the current applications of others. On the one hand, unlike DA algorithm and similar to the IA algorithm, after students select their applications, the assignment of the seats is final. On the other hand, unlike IA and similar to DA, AA coordinates the application strategies of students. AA differs from IA even more substantially since in each round of AA students can update their information on the remaining students and seats and thus find a more suitable application strategy.

These properties of AA become most visible in the two extreme versions of the school choice problem: if either all schools have strict priorities, or, on the contrary, if schools are indifferent between all students. If priorities are strict, AA coincides with DA (Proposition 1.5). Thus, in this case AA is strategy-proof and stable and its outcome dominates all equilibrium outcomes of the IA game.

In the opposite extreme case, when schools are indifferent over all students the efficiency comparison is also in favor of AA. Similar to DA with single tie-breaking, AA always induces an ex-post efficient assignment (Proposition 1.1), while some equilibria of IA are not ex-post efficient (Example 1.6 in the Appendix). From the ex-ante perspective, AA is preferred over DA in that any efficiency loss in AA implies a loss in DA, but not vice-versa (Proposition 1.2 and Example 1.2).

If not only schools are indifferent, but also students have common ordinal preferences, we show that AA ex-ante dominates DA (Proposition 1.3). We also show that in this case AA, unlike IA, is approximately strategy-proof (Proposition 1.4). Based on these results we motivate the conjecture that AA is strategy-proofness in the general case as well.

The simulations of the general case with intermediately coarse priorities shows that a simplified version of AA welfare dominates DA. We find that for some specifications, this version of AA restores up to approximately one half of the welfare loss associated with DA.

Overall, the results presented in this paper show that AA is a reasonable alternative for either DA or IA to use in the school choice programs. The question, however, remains whether AA can be easily implemented in real-life as that would require all students to report their vNM preferences for all schools. Although this question is out of the scope of this paper, we can briefly mention our ideas on this issue.

One way to implement AA would be to help students and parents to form, understand and express their vNM utilities. This can be done in a standard way by binary comparison of lotteries over certain schools. Alternatively, one could elicit "true" cardinal preferences using hypothetical questions that involve some continuous good. For example, one could ask parents how long they would agree to queue in order to get to a particular school as opposed to some other school. Assuming the preferences are quasilinear in time and consistent between schools, we elicit the required vNM utilities. Similarly, one can also ask hypothetical questions that involve tuition fees and distance to schools.

Another way to implement AA is to use it as an add-on for the status-quo ordinal rules—DA or IA—by letting students report their vNM preferences voluntarily. If the school choice program uses DA, then AA algorithm with partial voluntary reporting of some students can help to redesign the tie-breaking rule for these students in the manner of signals as in Abdulkadiroğlu, Che, and Yasuda (2015). For instance, in Example 1.1 it is enough for either student 2 or student 3 to report her cardinal preferences to arrive to the ex-ante efficient allocation.

Alternatively, if the current rule is IA, then the rule designer can play on behalf of these students as is done in AA, while treating the ordinal reports of other students as their strategies. Again, in Example 1.1, any subset of students can continue playing their unique IA equilibrium strategy, but if either student 2 or student 3 reports her vNM utilities, she can avoid the harmful miscoordination.

These and other implementation questions are of high interest and importance and require further theoretical and experimental research.

1.8 Appendix: Proofs

Example 1.6. Let there be eight students, three schools with one seat each and another school with five seats. Schools are indifferent between students, and students have the following vNM utilities:

	1	2	3	4	5	6	7	8
s_1 :	6	4	2	2	2	2	2	2
s_2 :	4	6	2	2	2	2	2	2
s_3 :	0	0	6	6	6	6	6	6
s_4 :	0	0	0	0	0	0	0	0

.

The game induced by IA has an equilibrium in which students 1 and 2 rank their second schools as their top choices, while all other students top-rank their best school s_3. Similar to Example 1.5, this report results in the assignment of students 1 and 2 — $(1, s_2), (2, s_1)$ — that is dominated by $(1, s_1), (2, s_2)$. (The latter assignment is induced by both DA and AA.)

Proof of Lemma 1.1.

Proof. We need to show that in each round the sequence of steps converges. For a contradiction, assume that for some problem $(u, \succsim_S)$ and for some choice rule C the steps in some round r do not converge. Therefore, since the number of possible applications is finite, there is a loop of switchings w starting with some step t.

Given the loop of switchings w we construct a *causal* cycle $\omega = (i_1, ..., i_l, i_{l+1}, ...)$, in which each student i_l's switch at some step t *causes* the switching of the next student i_{l+1} in the cycle at some later step. We distinguish two types of this causality. In the loop w each student i switching from s to s' changes the environment in two ways: i weakly *increases* the assignment probability of at least one other student in $I \backslash \{i, a(s)\}$ at s (we do not account for students in $a(s)$ since i weakly decreases their incentive to switch from s) and weakly *decreases* the assignment probability of at least one other student at s' among students in $a(s')$. At least one student (w.l.o.g. let us pick the same student i) in the loop w strictly increases or strictly decreases the corresponding assignment probability (otherwise students are never affected by switchings of other students and there is no cycle and therefore no loop). If i strictly increases the assignment probability at s for some student in $I \backslash \{i, a(s)\}$ we say that i *invites* to s; if i strictly decreases the assignment probability at s' we say that i *pushes* students from s'. Analogously, if after student i switches from s to s', some student j prefers to switch to s — we say j is invited to s and then j follows i in some casual cycle; and if some student j' wants to switch from s' — we say j' is pushed from s' and student j' follows i in some casual cycle.

Due to the priority structure $\succsim_S$, a student j who was invited by i to s cannot by her switching push students from s. Similarly, a student j' that was pushed by i from s' cannot invite students to s'. In other words, pushed students can only push and invited students can only invite. Therefore all causal cycles can be partitioned into pushing cycles and inviting cycles.

Each inviting cycle leads to an unambiguous Pareto improvement which cannot be sustained in a loop, therefore there must be at least one pushing cycle. Consider this pushing cycle. Each student can be pushed from some school s only by a student with at least as high a priority at s. Therefore there exists a pushing cycle ω^* in which all students have the same priority (the subset of students with higher priority form such a cycle). But for each school in ω^* each student pushes some other student of the same priority and therefore the expected utility at each school unambiguously increases, but this cannot be sustained in a loop — contradiction.

$\square$

Proof of Lemma 1.2.

Proof. We need to show that for each problem and each choice function C, for each round r there exists $x^* \in X^r$ such that $x(AA(\cdot|C, x^*)) = x^*$. □

Proof of Proposition 1.1.

Proof. Observe first that AA leaves no desired seat empty: if that was the case and one student preferred an unassigned seat to the one she received then she should have applied to that seat in the round in which she got her assignment.

It is left to show that no group of students prefers to exchange their assigned seats among themselves. If that was the case, these students must have received their seats in the same round of AA (otherwise there is at least one student who desires a seat that was still available while he applied to another school, which contradicts the AA algorithm). Let us focus on this round.

In this round all students switch unilaterally. If a student switches from a more desired school s to a less desired school s' that can only be because s' promises a higher assignment probability. As schools are indifferent between students, the assignment probability at a given school is the same for each student applying at that school, and there can be no student i' with opposing preferences $u_{i's'} > u_{i's}$ who prefers to switch to s instead of s'. Thus, there are no two students who prefer to exchange their schools.

We can use a similar argument for a trading cycle of any size. For a contradiction, assume that at the end of one round there is a group of student-school applications $\{(i_1, s_1), ..., (i_n, s_n)\}$ which forms a trading cycle: for each $k = 1, ..., (n-1)$ it holds that $s_{k+1} \succ_{i_k} s_k$ and also $s_1 \succ_{i_n} s_n$. Among all students in the group, let i_1 be the last who ended up applying to s_1 after others converged to their final applications. Student i_1 could apply to s_1 only because the assignment probability at s_1 was strictly higher than at s_2. However, from the i_2's perspective, the assignment probability at s_2 is strictly higher than at s_3. Continuing this argument along the trading cycle, we get a set of mutually contradicting inequalities for assignment probabilities.

□

Proof of Proposition 1.2.

Proof. We show the result for sd-efficiency, the argument for ex-ante efficiency is identical.

If for a given problem AA is not sd-efficient, then there exists a group of two or more students such that for certain different tie-breaking lotteries (in all rounds), these students receive seats in schools that form a trading cycle (as in the proof of Proposition 1.1). Based on these tie-breaking lotteries, we can design new tie-breaking lotteries for DA such that each student in the group gets the same school and therefore the same cycle is formed.

Consider some student i from the group of students who received a school s in the AA outcome given the set of tie-breaking lotteries τ. Let us now define a new tie-breaking rule τ' such that i gets s under DA with this tie-breaking rule.

Under AA i did not get his most preferred school s_1 either because he applied and lost in τ — then we make i lose to the same student in τ', or i did not even apply to s_1 because the assignment probability was too low and i preferred to switch to another school — then we make i lose in τ' to just one student who won s_1 in τ under AA. (We do the same for all other students so that the set of applications under DA is consistent with AA.) We continue in the same way along the i's preferences until i applies at s and wins it in τ', and it completes the proof.

□

Proof of Proposition 1.3.

Proof. The logic of the proof is the following: for each type of student in the AA algorithm, instead of submitting his preferences truthfully, we let the student mix over all possible preferences and thus play an "average" strategy. We show that this average strategy delivers the same expected payoff as DA. Since AA chose the equilibrium strategy, it dominates the average strategy and thus dominates DA.

Let each school s have a unit capacity: $\kappa_s = 1$; and let the number of schools be equal to the number of students: $|S| = |I| \equiv N$. We first find the expected assignment probability under AA for any given type $u \in U$. By $P_s^{AA}(u)$ we denote the probability with which some student i of type u_i receives a seat in s for *any* set of preferences of other students:

$P_s^{AA}(u_i) = \sum_{(u_i,u_{-i})\in\mathcal{U}} \varphi_{is}^{AA}((u_i, u_{-i})f(u_{-i}),$

where φ_{is}^{AA} denotes i's assignment probability at s under AA and $f(u_{-i})$ is the distribution of types u_{-i}: $f(u_{-i}) = \prod_{u\in u_{-i}} f(u)$.

If we sum up $P_s^{AA}(u_i)$ for all types $u_i \in U$, since in our setting there are just as many seats as students and in the AA outcome no seat remains unassigned, we find the assignment probability for each school s:

$\sum_{u_i \in \mathbf{u}} N P_s^{AA}(u_i) f(u_i) = 1.$

The RHS is just the unit capacity at each school s, while the LHS is the total expected number of students getting a seat in s: there are N students in total and each of them draws type u_i with probability $f(u_i)$, and then under AA he gets a seat at school s with probability P_s^{AA}.

Let us now design the average reporting strategy for some student i with type u_i. Instead of reporting u_i let i mix all possible types with objective weights: i reports to be of some type $u_j \in U$ with probability $f(u_j)$. Then i expects to get a seat at school s with the following probability:

$\sum_{u \in U} P_s^{AA}(u) f(u) = \frac{1}{N},$

where the equality follows from the previous statement. As under DA each student gets precisely $\frac{1}{N}$ of each seat, the average strategy delivers i the same expected payoff as DA. Since AA an equilibrium strategy for any preference of i given others students' applications, AA ex-ante dominates DA.

□

Proof of Proposition 1.4.

Proof. We first identify the manipulation channels and incentives for AA algorithm with the choice rule C^* and then show that these incentives disappears as the market grows large.

No student i can benefit from misreporting unless he manages to change the distribution of applications at the end of some round of the AA algorithm (since the distribution of applications the end of each round determines the expected payoff of each student). He can change the distribution of applications only by manipulating one of his λ-s so that he is picked before or after some other students. (Referred to as a non-bossy case in the motivation of Conjecture 1.)

Student i's incentive to misreport comes from the expected payoff from his application at the end of the round (compared to the expected payoff if he reported truthfully), which depends on the number of applicants competing with i for the same school. Choice rule C^* is non-bossy: no student can change the distribution of applications at the end of a round without changing his or her own application.

Besides, since all students have the same priority at all schools, each student can change the number of applicants at a particular school at most by one.

Let us now focus on the benefits from manipulation. Let student i apply to school s at the end of the round if he reports truthfully and to school s' if i switches. If i prefers s' to s then he switches later then he would otherwise, and pushes some student to switch from s', we call it a trigger-strategy. If, on the other hand, i prefers s to s' then he prefers to switch earlier than some other students and he deters one student from switching to s', we call a deterrence-strategy.

What happens now as the market grows large? Namely, let all students and all seats be replicated m times. As the switching procedure is deterministic, if i and all its replicas *can* coordinate and manipulate their report in the same way as before, then each of them has the same expected gain as i had in the original small market. However, as i *cannot* coordinate with its replicas, he can only pursue his manipulation strategy alone. If i pursued the deterrence-strategy (switching to s' prior to some student j, who then prefers not to follow i), in a large market i's replicas will keep applying to the same school and thus push j and its replicas to s'. As m grows large, the assignment probability that j and its replicas receive at s' converges to the assignment probability under truthful report of i and therefore for after high enough m, j and all her replicas apply to s' making i's manipulation suboptimal. The same is true for the trigger-strategy: in the large market all i's replicas will from s' and, when n is high enough, i alone will not trigger any student j to switch from s'.

Thus, AA is approximately strategy-proof.

□

Chapter 2

Efficient Lottery Design

This chapter is based on Kesten, Kurino and Nesterov (2015).

2.1 Introduction

A lottery is a common tool to establish fairness in real-life indivisible goods allocation problems such as object/task assignment, on-campus housing, kidney exchange, course allocation, and school choice. The simplest of these problems is the so-called assignment problem, where a set of distinct objects is allocated to a set of agents. A widely used real-life mechanism for such problems is the *random serial dictatorship* (RSD): a random ordering of agents is drawn from a uniform lottery, and the first agent picks her favorite object; the second agent picks her favorite object among the remaining ones; and so on. RSD satisfies many desirable properties. Ex post efficiency is an important one: after the resolution of the lottery, the resulting deterministic assignment is Pareto efficient. In a number of school districts, where schools are equipped with possibly distinct and coarse priority orders over students, popular assignment mechanisms such as Boston and Deferred Acceptance (Gale and Shapley, 1962) are applied upon randomly breaking the ties in schools' priority orders. All of these mechanisms, which we henceforth refer to as *lottery mechanisms*, induce a probability distribution over deterministic assignments, i.e., a lottery over mappings of agents to objects.

Notwithstanding the prominence and popular usage of lottery mechanisms in practice,[1] there has been much recent interest in *stochastic mechanisms* that prescribe the marginal probabilities with which each agent is assigned each object. In other words, a stochastic mechanism, unlike a lottery mechanism, does not immediately output a deterministic assignment but rather outputs a (sub)stochastic assignment matrix indicating agents' marginal assignment probabilities. To implement a stochastic mechanism one often resorts to a Birkhoff-von Neumann type of decomposition that transforms the outcome of the stochastic mechanism into an equivalent lottery over deterministic assignments. An important advantage and a chief motivation of the stochastic approach is that it makes it possible to achieve superior efficiency properties relative to lottery mechanisms. A well-known example of this appraoch is the *probabilistic serial* (PS) mechanism by Bogomolnaia and Moulin (2001) (hereafter BM),[2] which has become the cornerstone of a rapidly growing body of literature concerning stochastic mechanisms (cf. Che and Kojima, 2010; Kojima and Manea, 2010; Hashimoto et al., 2014).

BM have pointed out that the RSD outcome may suffer from unambiguous efficiency losses regardless of the von Neumann-Morgenstern utilities compatible with agents' ordinal preferences. Manea (2009) shows that these losses are prevalent even in large assignment problems. BM introduce a stronger notion of efficiency, which we call "sd-efficiency": a stochastic assignment is *sd-efficient* if it is not dominated by another stochastic assignment. Surprisingly, RSD may not always induce sd-efficient outcomes. BM have proposed PS as a serious contender to RSD, which selects the central point within the sd-efficient set. The attractive sd-efficiency (as well as the sd-envy-freeness) property has triggered much interest to further extend and generalize PS to richer and more structured assignment problems (cf. Kojima 2009; Athanassoglou and Sethuraman 2011; Budish et al. 2013).

[1] Indeed we are not aware of any stochastic mechanisms in use for any practical assignment problem.

[2] PS treats each object as a continuum of probability shares and allows agents to simulatneously "eat away" from their favorite objects at the same speed until each agent has eaten a total of 1 probability share. The share of an object an agent has eaten during the process represents the probability with which she assigned the object by PS. See Section 5 for a more precise decription.

An obvious advantage of lottery mechanisms is that they largely facilitate ex post analysis, which may focus on considerations such as incentives, fairness, stability, individual rationality, and efficiency. Nevertheless, the lottery approach has not been as successful as the stochastic approach as far as achieving stronger welfare properties than ex post efficiency.[3] Nevertheless, because a stochastic assignment needs to be decomposed into a feasible lottery before actual implementation (Birkhoff, 1946; von Neumann, 1953; Kojima and Manea, 2010), ex post considerations are comparably more difficult, if not impossible, to handle in the domain of stochastic assignments.[4] Therefore, we believe that bridging the gap between the two approaches and developing tools that would allow one to work directly with lotteries without sacrificing efficiency is an important task. In this paper, our goal is to show that ex ante efficiency analysis in addition to ex post analysis can be performed directly using lotteries.

We set off on our quest by uncovering the link between ex post efficiency and sd-efficiency. In a related paper, Abdulkadiroğlu and Sönmez (2003) study whether the sd-inefficiency of a stochastic assignment could be attributed to the Pareto inefficiency of a deterministic assignment it may induce and give a negative answer to this question. We provide a complementary result to this observation. In particular, we show that for any given stochastic assignment P of any given assignment problem $\succ$, there exists a corresponding deterministic assignment $\mu(P, \succ)$ that is Pareto efficient if and only if P is sd-efficient at $\succ$ (Theorem 2.1). The deterministic assignment $\mu(P, \succ)$ is obtained by transforming the $n-$agent stochastic assignment problem into an at most n^2-agent deterministic assignment problem that introduces multiple replicas of each agent. An immediate corollary is Abdulkadiroğlu and Sönmez's characterization of sd-efficiency via notions of domination across sets of assignments.

[3]For example, as far as we are aware, a nontrivial lottery mechanism satisfying sd-efficiency (or the stronger ex-ante efficiency) is yet to be reported or studied. Additionally imposing strategy-proofness readily leads to impossibilities (Zhou, 1990; Bogomolnaia and Moulin, 2001).

[4]Budish et al. (2013) develop tools for handling complex constraints while working directly with stochastic mechanisms.

An important contribution of our study, in line with the commonly used methodology and trends in indivisible goods allocation literature,[5] is to develop methods for the construction of a lottery that improves upon a given inefficient lottery while maintaining the feasibility of the final outcome (Theorem 2.2).[6] We observe, however, that the former part of such an objective may turn out to be quite subtle, as an ex ante welfare improvement over an ex-post lottery can actually give rise to an ex-post inefficient lottery (Example 2.1). For the latter part of the objective, we propose an algorithm that generates a feasible lottery from an infeasible lottery provided that it has a feasible equivalent. As an application of our tools and ideas, we propose new lottery mechanisms that stochastically improve upon RSD. Our proposals combine the above-mentioned methods with the celebrated object assignment method called the *top trading cycles (TTC)* method, attributed to David Gale. One of these proposals, which we call the TTC-based RSD (TRSD) mechanism, is sd-efficienct, stochastically dominates RSD, and satisfies equal treatment of equals (Theorem 2.3).

Finally, we offer a lottery representation of PS for any given problem. The idea is based on the identification of a set of priority orders such that the equal-weight lottery over the serial dictatorship outcomes induced by the collection of these priority orders results in exactly the same stochastic assignment as the PS outcome. Recall that RSD is an equal-weight lottery over all possible priority orders of agents regardless of agents' preferences. Unlike the RSD lottery, however, the set of priority orders in the support of the lottery representation of PS, is constructed based on agents' preferences. This implies that to implement PS as a lottery mechanism, we need to elicit agents' preferences a priori and determine the set of priority orders to be used in the lottery draw. Once the support of the lottery is constructed, the rest

[5]Improving upon a "status quo" allocation (or a partial allocation) while respecting other considerations has been a common goal in various applications of indivisible goods allocation. Examples of applications include housing markets, on-campus housing, kidney exchange, and school choice. All these applications, however, have focused on achieving ex post properties.

[6]In a related paper, Manea (2008) shows the existence of lotteries that improve upon the RSD outcome. Differently than here, his approach is based on working directly with stochastic assignments.

of the assignment process proceeds in exactly the same way as with RSD: the first agent picks her favorite object; the second agent picks her favorite object among the remaining agents; and so on. We generalize this approach by proposing a lottery representation algorithm that, for any given stochastic assignment, generates an equivalent equal-weight lottery (Theorem 2.4).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 establishes a link between ex post and ex ante efficiency and describes our algorithm for generating a feasible lottery. Section 4 introduces the TTC-based RSD mechanisms and Section 5 the lottery representation of PS. Secion 6 concludes.

2.2 The Model

A discrete resource allocation problem is a list $(N, A, q, \succ)$ where $N = \{1, \ldots, n\}$ is a finite set of agents; A is a finite set of objects; and $q := (q_a)_{a\in A}$ is a positive integer vector where q_a denotes the **quota** of object $a \in A$. We assume that $|N| \leq \sum_{a\in A} q_a$; $\succ = (\succ_i)_{i\in N}$ is a preference profile where $\succ_i$ is the strict preference relation of agent $i \in N$ on A. Let $\succeq_i$ denote the weak relation associated with $\succ_i$. The null object, if assumed to exist, is an object in A denoted by a_0, which is assigned a quota of n so that all agents can simultaneously consume it. Agents who are assigned the null object are viewed as taking their outside options. We fix N, A, and q throughout the paper, and denote a problem by a preference profile $\succ$.

A **(deterministic) assignment** is a function $\mu : N \to A$. Moreover, it is **feasible** if for each $a \in A$, $|\mu^{-1}(a)| \leq q_a$. Let $\mathcal{D}$ be the set of all assignments, and let $\mathcal{D}^f$ be the set of all feasible assignments. A feasible assignment μ is **Pareto efficient** at $\succ$ if there is no $\mu' \in \mathcal{D}^f$ such that for all $i \in N$, $\mu'(i) \succeq_i \mu(i)$, and for some $i \in N$, $\mu'(i) \succ_i \mu(i)$. A **deterministic mechanism** associates a feasible assignment with each problem.

A **stochastic allotment** is a probability distribution $P_i := (p_{i,a})_{a\in A}$ over A where $p_{i,a}$ denotes the probability that agent i receives object a, and thus for each $a \in A$, $0 \leq p_{i,a} \leq 1$ and $\sum_{b\in A} p_{i,b} = 1$. A **stochastic assignment** $P = [P_i]_{i\in N} = [p_{i,a}]_{i\in N,\ a\in A}$ is a substochastic matrix such that for each $i \in N$

and each $a \in A$, $\sum_{b \in A} p_{i,b} = 1$ and $\sum_{j \in N} p_{j,a} \leq q_a$. Let $\mathcal{S}$ be the set of all stochastic assignments. A **stochastic mechanism** associates a stochastic assignment with each problem.

Definition 2.1. A **lottery** $L = \sum_{s \in S} w_s \mu_s$ is a probability distribution over assignments such that

(L1) set S, called an **index set**, is nonempty and finite;

(L2) $\sum_{s \in S} w_s = 1$;

(L3) for each $s \in S$, $0 < w_s \leq 1$ and w_s is a rational number; and

(L4) for each $s \in S$, $\mu_s \in \mathcal{D}$,

where w_s is called the **weight** of μ_s, and $\mu_S = (\mu_s)_{s \in S} \in \mathcal{D}^S$ is the **support** of L. Moreover, it has **equal weights** if for each $s \in S$, $w_s = 1/|S|$ and it is **feasible,** if instead of (L4), it satisfies (L4'): for each $s \in S$, $\mu_s \in \mathcal{D}^f$.

Note that the support is a product set, contrary to the standard terms.[7] Also note that the index set is finite and the weights are rational numbers.[8] A **(feasible) lottery mechanism** associates a (feasible) lottery with each problem.

For each assignment $\mu \in \mathcal{D}$, let $\pi(\mu)$ be a $|N| \times |A|$ matrix that represents μ. Note that a given feasible lottery $L = \sum_s w_s \mu_s$ induces the stochastic assignment $P = \sum_s w_s \pi(\mu_s)$. Therefore, every feasible lottery mechanism can be uniquely represented as a stochastic mechanism. Given any stochastic assignment, the well-known Birkhoff-von Neumann theorem states that there is at least one feasible lottery that induces it. However, a stochastic mechanism may not be uniquely represented as a feasible lottery mechanism.

We say that two lotteries are **equivalent** if they induce the same stochastic assignment. The following is a useful lemma.

Lemma 2.1. *For each lottery, there is an equivalent equal-weight lottery.*

This result follows from duplicating assignments and expanding the original index set. See the Appendix for the proof.

[7] The reason for this will be clear when relating the sd-efficiency of a lottery with the Pareto efficiency of an assignment in a replica economy in the next section.

[8] This tractability assumption holds generally in practice and is satisfied by lotteries induced by all well-known mechanisms.

2.2.1 The random serial dictatorship mechanism (RSD)

We introduce a popular lottery mechanism, called the random serial dictatorship, which will be our focus in this paper. To this end we use a **priority** of agents in N that is a bijection from $\{1, 2, \dots, |N|\}$ to N. For example, given a priority f, $f(1)$ is the agent with the highest priority, $f(2)$ is the one with the second-highest priority, and so on. Let F be the set of all priorities.

Next is the **serial dictatorship (deterministic) mechanism induced by a priority** $f \in F$. We denote it by SD_f. Fix a problem $\succ$. The assignment $SD_f(\succ)$ is found interatively as follows.

Step 1: The highest priority agent $f(1)$ is assigned her top-choice object under $\succ_{f(1)}$.

$\vdots$

Step k: The kth highest priority agent $f(k)$ is assigned her top-choice object under $\succ_{f(k)}$ among the remaining objects.

Now we are ready to define the **random serial dictatorship mechanism (RSD)**, denoted by RSD: Fix a problem $\succ$. First, a priority f is chosen with probability $1/n!$. Second, agents are assigned objects according to $SD_f(\succ)$. Formally,

$$RSD(\succ) = \frac{1}{n!} \sum_{f \in F} SD_f(\succ).$$

Note that RSD is a lottery mechanism and its index set is the set F of all priorities.

2.2.2 The probabilistic serial mechanism (PS)

For each problem $\succ$, the stochasic assignment of the **probabilistic serial mechanism (PS)** is computed via the following simultaneous eating algorithm:[9] Given a problem $\succ$, think of each object a as an infinitely divisible good with supply q_a that agents eat in the time interval $[0, 1]$.

Step 1: Each agent eats away from her top-choice object at the same unit speed. Proceed to the next step when some object is completely exhausted.

[9]See Hugh-Jones et al. (2014) for an experimental evaluation of PS.

$\vdots$

Step k: Each agent eats away from her top-choice object from her remaining ones at the same unit speed. Proceed to the next step when some object is completely exhausted.

The algorithm terminates after some step when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The stochastic allotment of an agent i by PS is then given by the amount of each object she has eaten until the algorithm terminates. Let $PS(\succ)$ be the stochastic assignment of PS for problem $\succ$.

2.2.3 Axioms

A feasible lottery is **ex-post efficient** if it can be represented as a probability distribution over Pareto-efficient feasible assignments. BM propose an appealing ex ante notion of sd-efficiency that also implies ex post efficiency, which we introduce next. Fix a problem $\succ$. Given $i \in N$ and $P, R \in \mathcal{S}$, P_i **stochastically dominates** R_i at $\succ_i$ if for each $a \in A$, $\sum_{b \in A: b \succeq_i a} p_{i,b} \geq \sum_{b \in A: b \succeq_i a} r_{i,b}$. In addition, P **weakly stochastically dominates** R at $\succ$ if for each $i \in N$, P_i stochastically dominates R_i at $\succ_i$. P **stochastically dominates** R at $\succ$ if P weakly stochastically dominates R at $\succ$ and $P \neq R$. A stochastic assignment is **sd-efficient** at $\succ$ if it is not stochastically dominated by another stochastic assignment at $\succ$. Next is a much weaker efficiency property. A stochastic assignment $P \in \mathcal{S}$ is **non-wasteful** at $\succ$ if for each $i \in N$, each $a \in A$ with $p_{i,a} > 0$, and each $b \in A$ with $b \succ_i a$, we have $\sum_{j \in N} p_{j,b} = q_b$. Sd-efficiency implies ex post efficiency and non-wastefulness, but not vice versa.

We define our fairness axiom. A stochasitc assignment $P \in \mathcal{S}$ satisfies the **equal treatment of equals** at $\succ$ if for all $i, j \in N$, $\succ_i = \succ_j$ implies $P_i = P_j$.

Axioms of a lottery mechanism except ex post efficiency are defined for its induced stochasic assignment for each preference profile. A stochastic (lottery) mechanism is said to satisfy a property if for each preference profile, its (induced) stochastic assignment satisfies that property.

A stochastic mechanism φ is **sd-strategy-proof** if for each problem

$\succ$, each $i \in N$, and each preference $\succ_i$, $\varphi_i(\succ)$ stochastically dominates $\varphi_i(\succ_i', \succ_{-i})$ at $\succ_i$. A lottery mechanism is sd-strategy-proof if its induced stochastic mechanism is sd-strategy-proof.

A **stochastic mechanism** φ **weakly stochastically dominates a stochastic mechanism** ψ if for each problem $\succ$, $\varphi(\succ)$ weakly stochastically dominates $\psi(\succ)$. Moreover, a **stochastic mechanism** φ **stochastically dominates a stochastic mechanism** ψ if φ weakly stochastically dominates ψ and for some problem $\succ$, $\varphi(\succ)$ stochastically dominates $\psi(\succ)$ at $\succ$. Similarly, we can define the stochastic dominance of a lottery mechanism by looking at its induced stochastic mechanism.

Remark 2.1. RSD is known to be sd-strategy-proof, ex-post efficient, and to satisfy the equal treatment of equals. However, it is wasteful (Erdil, 2014) and thus is not sd-ecient (Bogomolnaia and Moulin, 2001). Moreover, PS is known to be sd-ecient and to satisfy the equal treatment of equals but not be sd-strategy-proof (Bogomolnaia and Moulin, 2001).

2.3 Sd-efficiency and Pareto efficiency

2.3.1 Characterization of sd-efficiency

Abdulkadiroğlu and Sönmez (2003, JET) investigate a possible link between sd-efficiency and Pareto efficiency. In particular, they ask whether the lack of sd-efficiency of a stochastic assignment (or equivalently, the sd-inefficiency of all lotteries it induces) can be associated with the lack of Pareto efficiency of a feasible assignment induced by it. They show that such a link between the two efficiency notions fails to exist: even if every feasible assignment in the support of every feasible lottery that induces a stochastic assignment is Pareto efficient, this may not be sufficient to guarantee the sd-efficiency of this feasible lottery. Our first objective is to recover the link between the two efficiency notions—albeit in a different sense—through an intuitive characterization result. We show that the sd-efficiency of a given feasible lottery is in fact implied by (and does imply) the Pareto efficiency of a "special" allocation constructed from the support of this feasible lottery. Before stating this result more precisely, we need the following definition.

Definition 2.2. Let $\succ$ be a problem and S be an index set. We rename N as the set of types. In the $|S|$**-fold replica problem**, for each type $i \in N$, there are $|S|$ agents; for each object $a \in A$, the quota is $q_a|S|$; for each type $i \in N$, all $|S|$ agents of that type share the common preferences $\succ_i$ on A. Let i_s be the agent of type i indexed by $s \in S$, $N_s = \{1_s, \cdots, i_s, \cdots, n_s\}$ be the set of all agents indexed by s, and $N_S := \cup_{s \in S} N_s$ be the set of all agents. We say that $\succ_{N_s} := (\succ_{i_s})_{i_s \in N_s}$ is the $s-$**replica problem**, and $\succ_S := (\succ_{N_s})_{s \in S}$ denotes the $|S|$-fold replica problem.

An $|S|-$**fold replica assignment** is a function $\nu_S : N_S \to A$ such that for each $a \in A$, $|\nu_S^{-1}(a)| \leq q_a|S|$. Let $\mathcal{D}_S$ be the set of all $|S|-$fold replica assignments. Given $\nu_S \in \mathcal{D}_S$ and $s \in S$, an $s-$**replica assignment** is a function $\nu_s : N_s \to A$ such that for each $i_s \in N_s$, $\nu_s(i_s) = \nu_S(i_s)$. Thus we denote $\nu_S = (\nu_s)_{s \in S}$. Note that the s-replica assignment ν_s from an $|S|$-fold replica assignment ν_S can be thought of as an assignment for the original problem $\succ$, but need not be feasible in the original. Thus we introduce the following. An $|S|-$fold replica assignment $\nu_S = (\nu_s)_{s \in S}$ is **feasible** if for each $s \in S$, $s-$replica assignment ν_s is feasible, i.e., for each $a \in A$, $|\nu_s^{-1}(a)| \leq q_a$.

Now we relate an $|S|-$fold replica assignment with the support of a lottery. Given a support $\mu_S = (\mu_s)_{s \in S}$ of a lottery, **the** $|S|-$**fold replica assignment induced by the support** μ_S is the $|S|-$fold replica assignment where for all $s \in S$, each agent $i_s \in N_s$ is assigned object $\mu_s(i_s)$. Conversely, given an $|S|-$fold replica assignment ν_S, the **support (of a lottery) induced by the** $|S|-$**fold replica assignment** ν_S is the support in which at each $s \in S$, each agent $i \in N$ is assigned object $\nu_s(i_s)$. Note that a lottery with induced support does not always induce a stochastic assignment. It does, however, if its weights are equal:

Lemma 2.2. *The equal-weight lottery with the support induced by an* $|S|-$*fold replica assignment produces a stochastic assignment.*

The proof is omitted as it is straightforward. By Lemma 2.2, from now on, unless confusion arises, the support of an equal-weight lottery is an $|S|-$fold replica assignment, and vice versa.

An $|S|-$fold replica assignment μ_S **Pareto dominates** an $|S|-$fold replica assignment μ'_S at $\succ_S$ if for all $i_s \in N_S$, $\mu_S(i_s) \succeq_i \mu'_S(i_s)$ and for some $i_s \in N_S$, $\mu_S(i_s) \succ_i \mu'_S(i_s)$. Also, an $|S|-$fold replica assignment is **Pareto efficient** at $\succ_S$ if it is not Pareto dominated by any other $|S|-$fold replica assignment. The following result relates the Pareto dominance of $|S|-$fold replica assignments with the stochastic dominance of the equal-weight lottery with induced support.

Lemma 2.3. *Let S be an index set, and μ_S, μ'_S be $|S|-$fold replica assignments. Suppose that μ_S Pareto dominates μ'_S at $\succ_S$. Then, the equal-weight lottery with support μ_S stochastically dominates the equal-weight lottery with support μ'_S at $\succ$.*

We omit the straightforward proof. The following result links the sd-efficiency of a (feasible or infeasible) lottery and the Pareto efficiency of its support in the $|S|-$fold replica problem.

Theorem 2.1. *Let $\succ$ be a problem and L a lottery with an index set S. Then, lottery L is sd-efficient at $\succ$ if and only if the support of L is Pareto efficient at $\succ_S$.*

The characterization of sd-efficiency given by Theorem 2.1 is quite intuitive. Theorem 2.1 also forms the basis of a practical test of sd-efficiency as it uses the standard notion of Pareto efficiency for the support of a lottery in its replica problem. Whereas determining whether a stochastic assignment is stochastically dominated or not may be difficult, checking for the Pareto efficiency of the support of a lottery is fairly straightforward by drawing on the top trading cycles (TTC) method, which we later describe.[10]

[10] Simply apply the TTC to the problem where the support of the lottery is interpreted as an extended housing market with endowments. Then the following is easy to show. The support of the lottery is Pareto efficient if and only if the TTC algorithm generates only self-cycles.

2.3.2 An alternative proof of an sd-efficiency characterization

Based on Theorem 2.1, we next provide an alternative proof of Abdulkadiroğlu and Sönmez's (2003) characterization of sd-efficiency. To this end, we introduce some notion: an $|S|-$fold replica assignment μ_S is **frequency equivalent** to an $|S|-$fold replica assignment ν_S if for each $a \in A$, $|\mu_S^{-1}(a)| = |\nu_S^{-1}(a)|$. Their characterization is based on the following notion of domination. For exposition without additional notation, we adapt their notion in our replica problem.

Definition 2.3. Given an index set S, a feasible $|S|$-fold replica assignment μ'_S **AS dominates** an $|S|$-fold replica assignment μ_S if

1. there is an $|S|$-fold replica assignment $\bar{\mu}_S$ that is fequency equivalent to μ'_S, and
2. there is a one-to-one function $f : S \to S$ such that

(a) for each $s \in S$, $\bar{\mu}_s$ Pareto dominates or is equal to μ_s at $\succ$ and

(b) there is $s \in S$ such that $\bar{\mu}_s$ Pareto dominates μ_s at $\succ$.

Corollary 2.1. *(Abdulkadiroğlu and Sönmez, 2003)* Given a problem $\succ$, let feasible lottery $L := \sum_{s \in S} w_s \mu_s$ be an arbitrary decomposition of a stochastic assignment P. P is sd-efficient at $\succ$ if and only if *for each $T \subseteq S$, $\mu_T = (\mu_t)_{t \in T}$ is AS undominated.*

The proof of Corollary 2.1 is immediate from the following lemma and Theorem 2.1. Our alternative proof has the advantage of being more transparent and shorter than the original proof of Abdulkadiroğlu and Sönmez (2003) as our argument involves only elemenatry application of standard notions of Pareto efficiency to replica problems.

Lemma 2.4. *Let $\succ$ be a problem and S be an index set. Then μ_S is Pareto undominated if and only if for each $T \subseteq S$, μ_T is AS undominated.*

Proof. We prove the contrapositive of each direction. ($\Leftarrow$): If ν_S Pareto dominates μ_S, then it is straightforward to see that ν_S AS dominates μ_S. ($\Rightarrow$): Suppose that for some $T \subseteq S$, some μ'_T AS domintates μ_T. Then there is an $|T|$-fold replica assignment $\bar{\mu}_T$ that is fequency equivalent to μ'_T; and there is a one-to-one function $f : T \to T$ such that (a) for each $s \in T$, $\bar{\mu}_s$ Pareto dominates or is equal to μ_s at $\succ$ and (b) there is $s \in T$ such that $\bar{\mu}_s$ Pareto dominates μ_s at $\succ$. Then $\bar{\mu}_T$ Pareto dominates μ_T in the $|T|$-fold replica problem. Define ν_S as for each $s \in S$, $\nu_s = \bar{\mu}_s$; otherwise $\nu_s = \mu_s$. Then ν_S Pareto dominates μ_S in the $|S|$-fold replica problem. □

2.3.3 Welfare improvement from an ex-post efficienty lottery

In later sections, we aim to show that ex ante efficiency analysis as well as ex post analysis can be performed directly using lotteries. But before doing so, we make a useful observation about a possible ex post welfare consequence of stochastically improving upon a given feasible lottery. The next example shows that an ex ante welfare improvement over an ex-post efficient feasible lottery may actually entail an ex-post *inefficient* lottery.

Example 2.1. (Ex ante welfare improvement over an ex-post efficient lottery results in an ex-post inefficient lottery)

Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$, and $q_a = q_b = q_c = q_d = 1$. Preferences are as follows.

$$\begin{array}{c|cccc} \succ_1 & a & b & c & d \\ \succ_2 & a & b & c & d \\ \succ_3 & b & a & d & c \\ \succ_4 & b & a & d & c \end{array}$$

Consider the following ex-post efficient lottery.

$$L = \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & d & c \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & d & b & a \end{pmatrix},$$

$$\text{and } \pi(L) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

Next consider the following feasible lottery.

$$L' = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & c & b & d \end{pmatrix}}_{\mu_1} + \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & b & d & a \end{pmatrix}}_{\mu_2},$$

$$\text{and } \pi(L') = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

Clearly, lottery L' stochastically dominates lottery L. However, the support of L' contains the Pareto inefficient assignment μ_2. Thus L' is not ex-post efficient. We can show that there is no other feasible lottery that induces the stochastic assignment $\pi(L')$. $\diamondsuit$

Given that sd-efficiency implies ex post efficiency, the observation in Example 2.1 is counterintuitive. It implies that ex post efficiency is not preserved under welfare improvements in stochastic assignments. One of our objectives in this paper is to develop a method for constructing a new feasible lottery that stochastically improves upon a given sd-inefficient feasible lottery L while also ensuring ex post efficiency. To this end, we first take an equal-weight lottery with suppot μ_S equivalent to L (Lemma 2.1), and then by correspondence of the support and $|S|-$fold replica assignment (Remark 2.3.1), we consider a Pareto improvement from μ_S in the $|S|-$fold replica problem. However, there is a complication in the approach of obtaining a stochastically improving lottery: even if the initial lottery is feasible, the resulting lottery induced by a Pareto improvement may not be feasible. Thus, in Section 2.3.4, we propose a method that transforms a given infeasible lottery into an equivalent feasible one, and then in Section 2.4.2, we introduce a method of Pareto improvement in the replica problem with endowments.

2.3.4 Feasible-assignment-generating (FAG) Algorithm

Given an equal-weight but infeasible lottery with support $\mu_S = (\mu_s)_{s\in S}$, we introduce an algorithm that generates an equivalent and feasible lottery.

Note that as we defined in Section 2.3.1, an $|S|-$fold replica assignment ν_S is **feasible** if for each $s \in S$ and each $a \in A$, $|\nu_s^{-1}(a)| \leq q_a$.

Feasible assignment generating (FAG) algorithm

Initialization. Given is an $|S|-$fold replica assignment $\mu_S = (\mu_s)_{s \in S}$. Without loss of generality, assume $S = \{1, 2, \ldots, |S|\}$. We focus on swapping objects in the set $\bar{A} := \left\{a \in A \mid |\mu_S^{-1}(a)| > 0\right\}$—those that are assigned under μ_s for some $s \in S$. For given $i \in N$ and $s \in S$, $\mu_s(i)$ is sometimes denoted by $\mu_S(s, i)$. We use both notations whenever convenient. Let $\mu_S(S, i) = \left\{\mu_S(s, i) \in \bar{A} \mid s \in S\right\}$ and $\mu_S(1, I) = \left\{\mu_S(1, i) \in \bar{A} \mid i \in I\right\}$. Given $O \subseteq \bar{A}$, let

$$
\begin{aligned}
B(O) &= \cup_{i \in N:\, \mu(1,i) \in O} \left\{\mu_S(S, i)\right\}, \\
B^t(O) &= \begin{cases} O & \text{if } t = 1, \\ B(B^{t-1}(O)) & \text{if } t \geq 2. \end{cases}
\end{aligned}
$$

Phase 1 (Swap path identification). Let $a \in \mu_S(1, |S|)$ such that $|\mu_1^{-1}(a)| > q_a$, i.e., object a is assigned more agents than its quota at μ_1 (if no such object exists, μ_1 is feasible and we are done.). Let $X = \left\{c \in \bar{A} \mid |\mu_1^{-1}(c)| \leq q_c - 1\right\}$, i.e., the set of objects that are only partially assigned to agents at μ_1 under μ_S. Check if $B^1(\{a\}) \cap X \neq \emptyset$; if not, check if $B^2(\{a\}) \cap X \neq \emptyset$; $\ldots$; and so on. Let $t \in \mathbb{N}$ be the smallest number such that $B^t(\{a\}) \cap X \neq \emptyset$. This procedure is well defined by the following claim (see the Appendix for the proof).

Claim 2.1. 1) $B^0(\{a\}) \subseteq B^1(\{a\}) \subseteq B^2(\{a\}) \subseteq \ldots$;

2) For each $t \in \{0\} \cup \mathbb{N}$, if $B^t(\{a\}) \cap X = \emptyset$, then $B^t(\{a\}) \subsetneq B^{t+1}(\{a\})$;

3) There is $t \in \{0\} \cup \mathbb{N}$ such that $B^t(\{a\}) \cap X \neq \emptyset$. Thus, $\{a\} \subsetneq B^1(\{a\}) \subsetneq \ldots \subsetneq B^t(\{a\})$.

Phase 2 (Execution of swaps). Phase 1 implies that there are $(t + 1)$, $t \geq 1$, distinct objects $b_0 := a, b_1, \ldots, b_t := x$ such that $b_1 \in B(\{b_0\})$,

$b_2 \in B(\{b_1\})$, ..., $b_t = x \in B(\{b_{t-1}\}) \cap X$. This implies that there are t distinct agents, $i_1, i_2, \ldots, i_t$, and corresponding indices, $k_{i_1}, k_{i_2}, \ldots, k_{i_t}$ such that $\mu_S(1, i_1) = b_0 = a$ and $\mu_S(k_{i_1}, i_1) = b_1$; $\mu_S(1, i_2) = b_1$ and $\mu_S(k_{i_2}, i_2) = b_2$; ...; $\mu_S(1, i_t) = b_{t-1}$ and $\mu_S(k_{i_t}, i_t) = b_t = x$. Next update the support μ_S by setting $\mu_S(1, i_1) := b_1$ and $\mu_S(k_{i_1}, i_1) := b_0 = a$; $\mu_S(1, i_2) := b_2$ and $\mu_S(k_{i_1}, i_2) := b_1$, ..., $\mu_S(1, i_t) := b_t$ and $\mu_S(k_{i_t}, i_t) := b_{t-1}$.

Iteration. Given the support μ_S, repeating Phases 1 & 2 at most $n-1$ times yields a new support μ_S^1 whose first index assignment, μ_1^1, is feasible. Thus, we have finalized the first index assignment. Next we obtain a new support μ_S^2, whose first index assignment coincides with that of μ_S^1, by iteratively applying Phases 1 & 2 to the subsupport obtained from μ_S^1 by restrictling to the assignments from 2 to $|S|$. Thus we have finalized the second index assignment. Continuing similarly the algorithm terminates once we have cleared indices 1 through $|S| - 1$. The final support $\mu_S^{|S|-1}$ consists of $|S|$ feasible assignments. Therefore, we obtain the following.

Proposition 2.1. *Given an $|S|-$fold replica assignment μ_S, the FAG algorithm produces a feasible $|S|-$fold replica assignment that is frequncy equivalent to μ_S.*

The following is a collorary of Lemma 2.1 and Proposition 2.1.

Corollary 2.2. *Given any infeasible lottery, there is an equivalent feasible lottery with equal weights.*

We call a stochastic assignment **rational** if all of its entries are rational numbers. Then we can straightforwardly represent a rational stochastic assignment by an equal-weight infeasible lottery. Thus, as a corollary of Proposition 2.1, we have

Corollary 2.3. *Any rational stochastic assignment can be expressed as a feasible equal-weight lottery that induces it.*

Remark 2.2. Note that Corollary 2.3 gives a version of Birkho (1946); von Neumann (1953) when the stochastic assignment is restricted to be rational.

Example 2.2. (Finding feasible assignment)

Let $N = \{1,2,3,4,5,6\}$ and $A = \{a,b,c,d,e,f\}$ such that all of the objects have the quota of 1. Consider the following support $\mu_S = (\mu_1, \mu_2, \mu_3)$.

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & a & b & c & d & e \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & c & c & d & f & e \end{pmatrix},$$

$$\text{and } \mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ b & b & d & e & f & f \end{pmatrix}.$$

Initialization. We first tabulate these assignments into a table:

$$\mu_S = \begin{pmatrix} \mathbf{a} & [a] & \mathbf{b} & [c] & \mathbf{d} & [e] \\ a & [c] & c & d & \mathbf{f} & e \\ \mathbf{b} & b & \mathbf{d} & [e] & f & [f] \end{pmatrix}.$$

Phase 1 (Swap path identification). Observe that object a is assigned to multiple agents at $\overline{\mu}_1$ although $q_a = 1$; and object f is not assigned to any agent at μ_1, i.e., $X = \{f\}$. We start with $B(\{a\}) = \{a,b,c\}$. Since $B(\{a\}) \cap X = \emptyset$, we proceed with $B^2(\{a\}) = B(\{a,b,c\}) = \{a,b,c,d,e\}$. Since $B^2(\{a\}) \cap X = \emptyset$, we proceed with $B^3(\{a\}) = B(\{a,b,c,d,e\}) = A \setminus \{a_0\}$. Since $B^3(\{a\}) \cap X = \{f\}$, we conclude that $t = 3$.

Phase 2 (Execution of swaps). From Phase 1 we easily obtain a set of four objects $\{b_0 = a, b_1 = b, b_2 = d, b_3 = f\}$ such that $b \in B(\{a\})$, $d \in B(\{b\})$, and $f \in B(\{d\})$. In particular, we obtain a corresponding set of three agents $\{1,3,5\}$ such that $\mu(1,1) = a$ and $\mu(3,1) = b$; $\mu(1,3) = b$ and $\mu(3,3) = d$; and $\mu(1,5) = d$ and $\mu(2,5) = f$. The agents and their assignmnets identified in this fashion are indicated in boldface in the above table. (Note that such agent and object sets may not be uniquely obtained. An alternative path from object a to f is indicated in brackets in the above table.) Next we execute the vertical swaps to update the table as follows:

$$\mu_S = \begin{pmatrix} \mathbf{b} & \mathbf{a} & \mathbf{d} & \mathbf{c} & \mathbf{f} & \mathbf{e} \\ a & c & c & d & d & e \\ a & b & b & e & f & f \end{pmatrix}.$$

Iteration. Observe that the first row of the updated table above induces a feasible assignment, which is indicated in boldface. So we next reapply Phases 1 & 2 to the remaining two rows. Then it is not very difficult to see that the remaining table contains two trivial vertical swaps involving agent 5 and either of agents 2 and 3 for swapping object c with b, and object d with f. The following is one possible final table whose three rows induce the feasibe assignments μ_1, μ_2, and μ_3 respectively.

$$\mu_S = \begin{pmatrix} b & a & d & c & f & e \\ a & b & c & d & f & e \\ a & c & b & e & d & f \end{pmatrix}.$$

$\diamondsuit$

2.4 Lottery Mechanisms Dominating the Random Serial Dictatorship Mechanism

The most widely used lottery mechanism in real-life markets is the random serial dictatorship mechanism (RSD). However, as BM pointed out, RSD is not sd-efficient but only ex-post efficient. In this section, we propose a method of improving upon RSD.

2.4.1 Efficient lottery construction (ELC) procedure

We shall propose a method, the **efficient lottery construction (ELC)** procedure, to directly construct an sd-efficient lottery that stochastically dominates a given equal-weight sd-inefficient lottery $L = \frac{1}{|S|}\sum_{s\in S}\mu_s$. For a given problem $\succ$, our procedure is as follows.

Stage 1 (Improvement). We consider the $|S|$-fold replica problem $\succ_S$ with endowments $\mu_S = (\mu_s)_{s\in S}$ where each agent $i_s \in N_S$ owns an object $\mu_s(i_s)$.

Note that because L is sd-inefficient, by Theorem 2.1, its support μ_S is Pareto inefficient in the replica problem. Then we apply a Pareto improvement algorithm (to be introduced in the next subsection), which selects a Pareto efficient assignment ν_S.

Stage 2 (FAG algorithm). We apply the FAG algorithm (Section 2.3.4) to obtain a feasible $|S|-$fold replica assignment ν_S^f.

Stage 3 (New lottery). Take the equal-weight lottery $L' := \frac{1}{|S|}\sum_{s\in S}\nu_s^f$.

Theorem 2.2. *For each problem $\succ$ and each feasible sd-inefficient lottery L, the ELC algorithm induces an sd-efficient lottery that stochastically dominates L.*

Proof. Because ν_S is Pareto efficient at the replica problem, by Theorem 2.1, the induced lottery is sd-efficient. Moreover, ν_S Pareto dominates μ_S, by Lemma 2.3, and lottery L dominates lottery L'. □

2.4.2 Top Trading Cycles (TTC) Algorithm

We introduce a Pareto-improving algorithm that we alluded to in the ELC procedure. This is based on the well-known idea of Gale's top trading cycles (Shapley and Scarf, 1974). The top trading cycles (TTC) algorithm was originally introduced for a housing market where each object is owned by only one agent.[11] In contrast, we deal with replica problems with endowments where an object is owned by multiple agents. For this reason, we introduce a priority $g \in F$ as if an object were owned by only the highest-priority owner.

For a given priority $g \in F$ and a given replica problem $\succ_S$ with endowments μ_S, the **TTC algorithm** induces an $|S|-$fold replica assignment as follows:

[11] Because of its appealing efficiency and incentive features, a number of mechanisms based on the TTC method have been proposed and characterized for a variety of applications such as on-campus housing, school choice, and kidney exchange. Although for deterministic settings, all proposed TTC based mechanisms are Pareto efficient, little is known about the applicability of this procedure to the stochastic assignment context or its relation to sd-efficiency, for that matter. An exception is Kesten (2009) who shows that if a simple version of the TTC method is applied to a market in which each agent is initially endowed with an equal probability share of each object, then the resulting outcome is sd-efficient and coincides with that of PS.

Step 0: For each object $a \in A$, assign a counter that keeps track of how many copies of the object are available. Initially set the counter equal to $q_a|S|$.
Step 1: Each agent $i_s \in N_S$ points to her favorite object according to $\succ_i$. Each object points to the highest-priority type among those who own the object according to priority g. If there are several agents of the same type, pick one of them arbitrarily. There is at least one cycle where a cycle is a finite list of objects and agents $(a^1, i^1, a^2, i^2, \ldots, a^m, i^m)$ such that each agent i^ℓ points to object a^ℓ ($\ell \in \{1, \ldots, m\}$), and agent i^m points to object a^1. Each agent in a cycle is assigned a copy of the object that she is pointing to and is removed. The counter of each object in the cycle is reduced by one, and if it reduces to zero, the object is also removed. Counters of all the other objects stay the same.
Step k: Each remaining agent i_s points to her favorite object among the remaining ones according to $\succ_i$. Each remaining object points to the highest-priority remaining type according to priority g. If there are several agents of the same type, pick one of them arbitrarily. There is at least one cycle. Each agent in a cycle is assigned a copy of the object that she is pointing to and is removed. The counter of each object in the cycle is reduced by one, and if it reduces to zero, the object is also removed. Counters of all the other objects stay the same.

The above algorithm terminates in a finite step when all agents are assigned objects.
Last step: Note that the assignment ν_S induced by the above algorithm is not always feasible in the sense that some s-replica assignment ν_s is not feasible in the original problem $\succ$. For this reason, we apply the FAG algorithm to obtain a feasible assignment, which we denote by $TTC_S(\succ_S, \mu_S, g)$.

Note that the TTC algorithm implements Stages 1 and 2 in the ELC procedure. Now we are ready to state Proposition 2.2 (the proof is omitted, as the idea is very similar to the one for the Shapley and Scarf's (1974) for the housing market):

Proposition 2.2. *For each $|S|-$fold replica problem $\succ$ with endowments μ_S, the TTC algorithm induces a Pareto efficient assignment at $\succ_S$ that Pareto dominates μ_S and is equal to μ_S when μ_S is Pareto efficient at $\succ_S$.*

2.4.3 TTC-based random serial dictatorship^K (TRSD^K)

Using the ELC procedure and the TTC algorithm discussed in Sections 2.4.1 and 2.4.2, we propose an ex-post efficient lottery mechanism that dominates RSD and satisfies the equal treatment of equals — what we call the TTC-based random serial dictatorship^K mechanism (TRSD^K) given a natural number $K \in \{1, \ldots, n!\}$.

Let us consider how to improve upon RSD. With our tools developed so far–in particular–Lemma 2.3, we need to convert the problem into a replica problem with endowments. Ideally it is best to take the set of priorities, F, as the index set for the replica problem. However, as the number of agents, n, becomes large, the size of F, $n!$, becomes huge and computationally difficult to work on. To avoid this problem, we pick only $|K|$ distinct priorities, $f_1, \ldots, f_K$, and then consider the improvement over the induced random serial dictatorship $\frac{1}{K}\sum_{k=1}^{K} SD_{f_k}(\succ)$ by using the improving method of the TTC discussed in the previous subsection. This is the key idea of our TRSD^K mechanism, which we introduce next.

Let a problem $\succ$ and $K \in \{1, \ldots, n!\}$ be given.

Step 1: We choose K distinct priorities $f_1, \ldots, f_K$ out of all $n!$ priorities with equal probability $1/\binom{n!}{K}$ where the set F of all priorities have $n!$ priorities, and $\binom{n!}{K}$ is the number of $K-$combinations from $n!$ elements. Let $F(K) := \{f_1, \ldots, f_K\}$.

Step 2: We consider an improvement of the lottery $\frac{1}{K}\sum_{k=1}^{K} SD_{f_k}(\succ)$ that is a lottery of choosing SD assignments $SD_{f_k}(\succ)$ with priority f_k being selected with equal priority $1/K$. Moreover, we choose a priority $g \in \{f_1, \ldots, f_K\}$ with equal probability $1/K$. Then we apply the TTC algorithm for the priority g to the problem $\succ_S$ with endowments $(SD_{f_k}(\succ))_{k=1}^{K}$, and then we obtain the $|K|$-fold replica assignment $TTC_{F(K)}(\succ_{F(K)}, SD_{F(K)}(\succ); g)$. Then we consider the induced equal-weight lottery $\frac{1}{K}\sum_{f \in F(K)} TTC_f(\succ_{F(K)}$ $, SD_{F(K)}(\succ); g)$.

We denote the resulting lottery by $TRSD^K(\succ)$, and can express it as

$$TRSD^K(\succ) = \frac{1}{|\mathcal{F}(K)|} \sum_{F(K) \in \mathcal{F}(K)} \frac{1}{K} \sum_{g \in F(K)} \frac{1}{K} \sum_{f \in F(K)} TTC_f(\succ_{F(K)}, SD_{F(K)}(\succ); g), \tag{2.4.1}$$

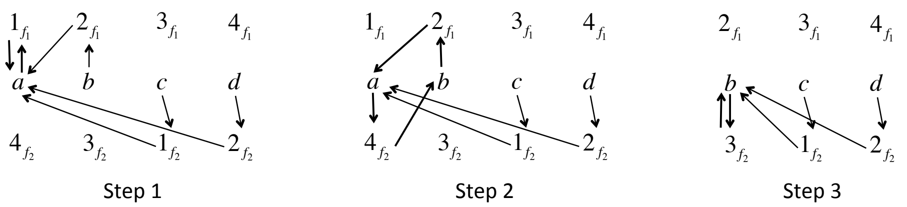
where $\mathcal{F}(K) := \{\{f_1, \ldots, f_K\} \mid f_1, \ldots, f_K \in F \text{ are distinct }\}$ and $|\mathcal{F}(K)| = \binom{n!}{K}$. Note that TRSD^1 coincides with RSD.

Example 2.3. We show how to implement $TRSD^K$ where $K \geq 2$. Consider $K = 2$ and an example where $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$, $q_a = q_b = q_c = q_d = 1$. Let preferences be given by:

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

.

Suppose that $f_1 = (1, 2, 3, 4)$ and $f_2 = (3, 4, 1, 2)$ are chosen and $g = f_1$. Then $SD_{f_1}(\succ) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$ and $SD_{f_2}(\succ) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ c & d & b & a \end{pmatrix}$. Then we apply the TTC algorithm as follows:



Here, for simplicity, we draw only the pointing arrows from agents who are also pointed at by objects, and we skip the remaining steps. We obtain $\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & a & c & d \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & d & b & b \end{pmatrix}$. Then, applying the FAG algorithm, we obtain $TTC_{f_1}(\succ_{F(K)}, SD_{F(K)}(\succ); f_1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & d & c & b \end{pmatrix}$ and $TTC_{f_2}(\succ_{F(K)}, SD_{F(K)}(\succ); f_1) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ c & a & b & d \end{pmatrix}$. One of these two assignments is selected with $1/2$ as a result of $TRSD^2$. $\Diamond$

Theorem 2.3. *Let $K \in \{2, \dots, n!\}$. TRSD^K is ex-post efficient, weakly stochastically dominates RSD, and satisfies the equal treatment of equals. Moreover, we have the following.*[12]

1. *Suppose the unit quotas of all objects, i.e., for each $a \in A$, $q_a = 1$. If $|N| \leq 3$ then $TRSD^K = RSD$. If $|N| \geq 4$, then $TRSD^K$ stochastically dominates RSD.*

2. *If RSD is not sd-efficient for some N, A, and q, then there is $\bar{K} \leq |A|$ such that for each $K \geq \bar{K}$, $TRSD^K$ stochastically dominates RSD.*

3. *For some N, A, q, and K, $TRSD^K$ is not sd-efficient.*

Clearly, TRSD^K is more tractable and practical when K is small. Note that K can be as large as $|F| = n!$. Part (2) asserts that in the stanrdard model of BM, we can improve upon RSD by taking only $K = 2$. Moreover, Part (3) asserts that in general K can be a small number relative to $n!$ in order to improve upon RSD.

Proof. We first show the ex post efficiency. By Proposition 2.2, $TTC_{F(K)}(\succ_{F(K)}, SD_{F(K)}(\succ); g)$ is Pareto efficient in the K-fold replica problem. Thus $TTC_f(\succ_{F(K)}, SD_{F(K)}(\succ); g)$ is Pareto efficient at the original problem $\succ$. Hence $TRSD^K$ is ex-post efficient.

We next show that $TRSD^K$ weakly stochastically dominates RSD. RSD can be expressed as

$$RSD(\succ) = \frac{1}{n!}\sum_{f\in F} SD_f(\succ) = \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{F}(K)} \frac{1}{K} \sum_{g\in F(K)} \frac{1}{K} \sum_{f\in F(K)} SD_f(\succ) \quad (2.4.2)$$

[12] It is quite challenging to check whether or not the TRSD^K is sd-strategy-proof, for the following reasons. First, BM's and Nesterov's (2014) impossibility theorems show the incompatibility of sd-strategy-proofness, sd-efficiency, and equal treatment of equals for problems with unit quotas. Thus their results are not applicable since TRSD^K is not necessarily sd-efficient in general, and nor does our setting assume unit quotas. Second, we need at least four agents for the outcomes of RSD and TRSD^K to differ, which makes it cumbersome to calculate the stochastic assignments of TRSD^K.

for each problem $\succ$. We compare (2.4.1) with (2.4.2): for each $F(K) \in \mathcal{F}(K)$ and each $g \in F(K)$, by Proposition 2.2, $TTC_{F(K)}(\succ_{F(K)}, SD_{F(K)}; g)$ Pareto dominates or coincides with $SD_{F(K)}(\succ)$. Thus, by Lemma 2.3, TRSD^K weakly stochastically dominates RSD.

In the Appendix we prove the equal treatment of equals and the stochastic dominance in Parts (1) and (2). It remains to show Part (3) – the sd-inefficiency. Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, a_0\}$, $q_a = q_b = 1$, and $\succ$ such that for each $i \in \{1, 2\}$, $a \succ_i b \succ_i a_0$; and for each $i \in \{3, 4\}$, $b \succ_i a \succ_i a_0$. Here a_0 is the null object. The computational simulation gives us the following stochastic assignments.

RSD

	a	b	a_0
Agent 1	0.4167	0.0833	0.5000
Agent 2	0.4167	0.0833	0.5000
Agent 3	0.0833	0.4167	0.5000
Agent 4	0.0833	0.4167	0.5000

$TRSD^2$

	a	b	a_0
Agent 1	0.4312	0.0688	0.5000
Agent 2	0.4312	0.0688	0.5000
Agent 3	0.0688	0.4312	0.5000
Agent 4	0.0688	0.4312	0.5000

$TRSD^3$

	a	b	a_0
Agent 1	0.4417	0.0583	0.5000
Agent 2	0.4417	0.0583	0.5000
Agent 3	0.0583	0.4417	0.5000
Agent 4	0.0583	0.4417	0.5000

Then we can see the following assignment P stochastically dominates $TRSD^2$ and $TRSD^3$: for each $i \in \{1, 2\}$, $P_i = (0.5, 0, 0.5)$; for each $i \in \{3, 4\}$, $P_i = (0, 0.5, 0.5)$. $\square$

2.5 Lottery Representation of the Probabilistic Serial Mechanism

Motivated by the sd-inefficiency of RSD, BM introduced a central stochastic mechanism that achieves sd-efficiency—the probabilistic serial mechanism

(PS). However, since PS is not a lottery mechanism, it might be less tempting to implement in practice, as discussed in the Introduction. In this section, we offer an algorithm of representing a PS stochastic assignment by an equal-weight lottery. Specifically, for each problem $\succ$, we construct a collection of priorities $F^* := (f_j)_{j=1}^J$, such that

$$PS(\succ) = \frac{1}{|J|}\sum_{j=1}^{J} SD_{f_j}(\succ).$$

Note that

$$RSD(\succ) = \frac{1}{|F|}\sum_{f \in F} SD_f(\succ),$$

where F is the set of all priorities. The differences between the set F and the collection F^* are threefold: (i) F^* depends on preference profiles, (ii) F^* might contain fewer different priorities than F does, and (iii) F^* will usually contain several copies of some of the priorities.[13] Before we proceed to the algorithm, consider the following motivating example.

Example 2.4. Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$, and $q_a = q_b = q_c = q_d = 1$. Consider the following problem and its PS assignment

$$\begin{array}{c|cccc} \succ_1 & a & b & c & d \\ \succ_2 & a & b & d & c \\ \succ_3 & a & c & d & b \\ \succ_4 & a & d & c & b \end{array} \qquad PS(\succ) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

where in the eating algorithm of PS, object a is first exhausted at time $1/4$, then object b at time $3/4$, and then objects c, d last at the same time of 1.

If we try to construct the lottery and the corresponding collection F^* for $PS(\succ)$, we first see that in any possible priority the first-priority agent always receives object a under SD. Thus there are at least four different priorities in F^*, and each of them begins with one of the four agents. It is

[13] In fact, we can show a more general result in which any sd-efficient stochastic assignment (and not only PS) can be represented as an equal-weight lottery using the same algorithm.

logical to assume that the objects exhausted earlier in the algorithm should also be assigned earlier in the lottery representation. Let us assume that in each priority the objects are assigned in the following order: a, b, c, d.

Consider a priority f_1 such that agent 1 has the first priority. Since object b is split between agents 1 and 2, only agent 2 can have the second priority. Similarly, agents 3 and 4 follow and thus $f_1 = (1, 2, 3, 4)$. The same logic applies if we begin another priority f_2 with agent 2. Then the only feasible sequence is $f_2 = (2, 1, 3, 4)$.

However, if we begin the ordering with agent 3, it is not clear whether agent 1 or agent 2 should receive the second priority, since they both have positive probability for object b. But if agent 1 follows and gets object b, then in the next step there is no agent left to be assigned object c, since neither agent 2 nor agent 4 receive it in expectation. Therefore, the only feasible ordering starting with agent 3 is $f_3 = (3, 2, 1, 4)$. Similarly, $f_4 = (4, 1, 3, 2)$.

In total, we have only four feasible priorities. Moreover, since object a is split equally between all agents, the weights of these priorities are equal to $\frac{1}{4}$. Therefore $F^* = (f_j)_{j=1}^4$. $\diamondsuit$

We now use the intuition from Example 2.4 to construct a general algorithm in the following steps: First order the objects. Then determine the set of feasible priorities (in the example there were only four). Next find the corresponding maximum weights of these priorities in the resulting lottery.[14] Finally calculate the individual contribution of each specific priority in the equal-weight lottery.

We begin by ordering the objects. Consider a problem $\succ$ and a stochastic assignment $P = PS(\succ)$. Let us relabel the objects as $a_1, a_2, ..., a_k$, in the exhausting order (denoted as l_{ex}) in the eating algorithm of PS. When two or more objects are simultaneously exhausted we order them arbitrarily. The objects that have only been partially exhausted are put in the end of the ordering in some arbitrary order: $a_{k+1}, ..., a_{|A|}$. For each object a_j, let $E(a_j)$ be the set of agents who have eaten a_j: $E(a_j) = \{i \in N | p_{i,a_j} > 0\}$.[15]

[14] This step is missing in the example since all priorities have the same weight.

[15] For a general case of an arbitrary sd-efficient assignment, objects can also be relabeled according to the exhausting order, although the underlying eating algorithm proceeds not

Next, given P and l_{ex}, for each priority $f \in F$, we determine the maximum weight $m(f, P, l_{ex}) := min_{j<|N|} p_{f(j),a_j}$ — the minimum of the assignment probabilities in P that correspond to agents in f and objects in l_{ex} (where $f(j)$ denotes the agent that has j^{th} priority in f). We refer to this weight as the maximum, since the final weight of this priority f in the lottery will be no higher than $m(f, P, l_{ex})$. If, for instance, the first agent in f does not belong to $E(a_1)$ and thus $p_{f(1),a_1} = 0$, then the overall maximum weight of f is zero and f does not enter the final lottery.

Importantly, for each f such that $m(f, P, l_{ex}) > 0$ the order in which the objects are picked in a serial dictatorship SD_f coincides with l_{ex}. This property follows from two facts. First, if $m(f, P, l_{ex}) > 0$, then each agent $f(j)$ has a positive probability share $p_{f(j),a_j} > 0$ of object a_j. For example, first-ordered agent $f(1)$ has a positive probability share $p_{f(1),a_1} > 0$ of object a_1, second-ordered agent $f(2)$ has a positive probability share $p_{f(2),a_2} > 0$ of object a_2, and so on. Second, ordering l_{ex} gives the following hierarchy for objects: in the PS eating algorithm a higher object is exhausted (weakly) earlier. Thus, each agent with a positive share of a_1 top-ranks a_1 (otherwise a_1 cannot be exhausted first); each agent with a positive share of a_2 either top-ranks a_2 or, alternatively, top-ranks a_1 and ranks a_2 as second, and so on. Therefore, in a serial dictatorship agent $f(j)$ picks a_j, and the order in which the objects are picked coincides with l_{ex}. (The same argument holds for different definitions of l_{ex} discussed in footnote 15.)

Among all priorities $f \in F$ we (arbitrarily) pick one of the priorities with the *lowest positive* maximum weight and denote it as f_1: $f_1 \in \arg\min_f \{m(f, P, l_{ex}) \mid m(f, P, l_{ex}) > 0\}$. This priority f_1 enters the resulting lottery with weight $m_1 = m(f, P, l_{ex})$.

Having determined f_1 and m_1, we subtract the corresponding assignment

using constant eating speed functions but some other profile of eating speed functions (BM). Alternatively, we can order the objects using the following hierarchical procedure: at each step agents point to their most preferred object among the remaining objects and we choose the most popular object (choose one of them arbitrarily if there are several) to be the next in our order of objects. Intuitively, this ordering of objects is similar to the exhausting order in the eating algorithm: at each step j the agents in $E(a_j)$ prefer object a_j over the remaining objects. This is the key feature of the ordering l_{ex} in our lottery decomposition.

from the old assignment matrix (denoted as $P_1 = P$). This way we get the updated matrix $P_2 := P_1 - m_1 SD_{f_1}(\succ)$. We then repeat the previous two stages for this updated matrix P_2 and continue doing so until all the relevant priorities together with the corresponding weights are determined. Then, similar to Lemma 2.1, we turn the lottery into the equal-weight lottery. Meanwhile, the set of relevant priorities that we picked at each stage becomes the collection of priorities F^* that defines the equal-weight lottery.

We now formally define the algorithm for the special case when the number of agents is the same as the total amount of objects ($|N| = \sum_{a\in A} q_a$), which implies that the objects have the unit quotas (for each $a \in A$, $q_a = 1$), and P is a bistochastic matrix.

Definition 2.4. (Lottery representation algorithm) Given $|N| = \sum_{a\in A} q_a$ and $(\succ, P, l_{ex})$, the lottery representation algorithm constructs the collection of priorities F^* as follows.
<u>Stage 1:</u> Let $P_1 = P$. Calculate $m(f, P_1, l_{ex}) := min_{j<|N|} p_{f(j),a_j}$ for each priority $f \in F$. Among all the priorities, pick $f_1 \in \arg\min_{f\in F}\{m(f, P_1, l_{ex}) \mid m(f, P_1, l_{ex}) > 0\}$ – one with the lowest positive maximum weight, denote the corresponding weight as $m_1 = m(f_1, P_1, l_{ex})$.
<u>Stage j:</u> Update the probability matrix as $P_j := P_{j-1} - m_{j-1} SD_{f_{j-1}}(\succ)$. For matrix P_j, find $f_j \in \arg\min_{f\in F}\{m(f, P_j, l_{ex}) \mid m(f, P_j, l_{ex}) > 0\}$ – the priority with the lowest maximum weight. Denote $m_j = m(f_j, P_j, l_{ex})$.
<u>Final stage r:</u> The updated matrix becomes null, i.e., $P_r = P_{r-1} - m_{r-1} SD_{f_{r-1}}(\succ) = 0$.
Given $\{f_j, m_j\}_{j=1}^r$, we construct the required collection F^* by finding the least common multiple for all the inverted weights $\frac{1}{m_0}$ and including each of the corresponding priority f_j in F^* precisely $\frac{m_j}{m_0}$ times.

The following theorem shows that the proposed iterative procedure is always feasible and results in the equal-weight lottery equivalent to the initial stochastic assignment P.

Theorem 2.4. *Given $|N| = \sum_{a\in A} q_a$, for each problem $\succ$ and a stochastic assignment P containing only rational elements and which is sd-efficient at*

$\succ$, the lottery representation algorithm induces an equal-weight lottery that is equivalent to P.

Proof. We first show the feasibility of operations at all stages of the algorithm. We show (a) the existence of the lowest maximum weight and the corresponding priority at any stage of the procedure and (b) the feasibility of updating the stochastic assignment matrix, given that we found the lowest maximum weight at the preceding stage. Then we make sure that (c) the algorithm terminates and that the representation is correct.

(a) We first show by induction that at any stage $j < r$, the matrix P_j is quasi-bistochastic (its columns and rows sum up to the same positive number). The claim is correct for $P_1 = P$. Assume it also holds for P_{j-1}. Due to the Birkhoff-von Neumann theorem, P_{j-1} can be decomposed as a convex combination of assignments (note that the lowest weight in this convex combination is weakly lower than the lowest element in P_{j-1}). Each assignment μ corresponds to some priority f defined as follows: the agent matched with a_1 receives the first priority in f, the agent matched with a_2 receives the second priority in f, and so on along l_{ex}. All such priorities f have positive maximum weights $m(f, P_{j-1}, l_{ex})$, which we define as the minimum element in P_{j-1} among the elements that correspond to assignment $SD_f(\succ)$. Among those priorities we pick f_{j-1} – the priority with the lowest positive maximum weight m_{j-1}.

(b) Given f_{j-1} and m_{j-1}, we update the assignment matrix as $P_j = P_{j-1} - m_{j-1}SD_{f_{j-1}}(\succ)$. In doing so, we subtract a positive number that was smaller than the lowest positive element in P_{j-1} from precisely one element in each row and in each column of P_{j-1}; we do not subtract from zero elements (otherwise f_{j-1} is not feasible). Thus P_j remains quasi-bistochastic.

(c) At each stage of the algorithm, the updated stochastic assignment contains at least one more zero element. Therefore, the algorithm terminates in $r \leq (|N|^2 - |N|)$ stages, since at the last stage r the stochastic assignment matrix degenerates into a weighted assignment $SD_{f_r}(\succ)$. It is straightforward from the updating formula to check whether $\sum_{j=1}^{r} m_j SD_{f_j}(\succ) = P$. $\square$

Now we extend the algorithm to the case when there are fewer agents than objects: $|N| \leq \sum_{a \in A} q_a$. We use a simple trick: for each problem $\succ$ and each stochastic assignment P, we add the total of $(\sum_{a \in A} q_a - |N|)$ artificial agents. The preferences $\succ'$ of each artificial agent i' are such that he prefers the objects that were originally left in expectation, i.e., with the total assignment probabilities being less than one, to the objects that were consumed fully: $a_l \succ'_{i'} a_j$, where $j < k+1 \leq l$. The preferences of the normal agents remain as before in $\succ$. Since the total number of agents is now the same as the number of objects, the assignment probabilities for the artificial agents are such that the modified stochastic assignment matrix P' becomes bistochastic.

We then run the lottery representation algorithm for the triple $(\succ', P', l_{ex})$, where the preferences and the stochastic assignment matrix include the artificial agents, but the order of objects l_{ex} remains the same as for $(\succ, P)$. After we receive the collection of priorities F'^{*}, we take out all the artificial agents from each of the priorities. The agents that were below the artificial agents in some priority f' now get a higher slot.

It is easy to see that the result of the new lottery is precisely P. First, in P' the artificial agents consumed only those probability shares that were not taken by normal agents in P; given Theorem 2.4, the same holds for the lottery representation of P'. However, when we take some artificial agent i' out of some priority f', given the preferences of the artificial agents, each normal agent i that was below i' in f' receives the same object that he received before agent i' was taken out. Therefore, if we take out all the artificial agents in f', then the assignment of normal agents does not change, and neither does the weight of f' in the lottery.

2.6 Concluding Remarks

In this paper, we have introduced new tools that allow the designer to work directly with lotteries and enhance the efficiency properties of existing lottery mechanisms. Whereas the stochastic approach has already proved extremely useful in achieving superior welfare features over its lottery counterparts, coupling lottery-type assignment methods with the tools developed here may help close the gap between the two approaches while also benefiting from the practical appeal of lottery mechanisms.

Our analysis of the construction of ex post and sd-efficient lotteries lends itself to new interpretations of the workings of the prominent mechanisms RSD and PS. Abdulkadiroğlu and Sönmez (1998) show that the lottery produced by RSD is equivalent to a lottery constructed in the following way: Start from the inittial lottery that assigns an equal probability (namely, $\frac{1}{n!}$) to each feasible assignment, and apply the TTC algorithm to each feasible assignment in the support of the initial lottery and replace feasible assignment by the corresponding outcome of the algorithm. Since the TTC algorithm produces Pareto efficient feasible assignments, such a lottery is ex-post efficient but not sd-efficient (as is the one induced by RSD). Kesten (2009) shows that the stochastic assignment produced by PS is equivalent to a stochastic assignment constructed in the following way: Start from an initial stochastic assignment that endows each agent each object with the same probability (namely, $\frac{1}{n}$) and apply the TTC algorithm (that considers self and pairwise-cycles) in a way that allows each agent to trade assignment probabilities of her most-preferred object with *every* other agent who is endowed with a positive probability for this object. Our analysis indicates that the difference between RSD and PS derives from the way they choose the improvement cycles from among those induced by the support of the initial lottery. Whereas RSD considers only those top trading cycles induced by each feasible assignment in the support of the initial lottery individually, PS considers all the top trading cycles induced by all feasible assignments in the support of the initial lottery altogether.

In the United States, many school districts use centralized clearinghouses to determine student assignments to public schools (Abdulkadiroğlu and Sönmez, 2003). In school choice, each school has multiple capacity and is assigned a priority order of students by the school dictrict to be used while determining student assignments. In many school districts student priorities are typically coarse, giving rise to weak priority orders. As a consequence, school districts rely on lottery mechanisms that use randomization to generate strict priority orders by breaking the ties among equal-priority students via lottery draws. Although an assignment problem is a special school choice problem with each school having unit capacity and all students having equal priority for all schools, our analysis can begeneralized straightforwardly and adapted to school choice problems, and in particular, could be helpful in improving the ex ante efficiency of school choice lotteries (see Kesten and Ünver, 2015).

2.7 Appendix: Proofs

Proof of Lemma 2.1. Let $L = \sum_{s\in S} w_s\mu_s$ be a lottery. The lemma is obvious if S is a singleton. Thus, suppose not. Without loss of generality, let $S = \{1, \ldots, |S|\}$. By (L2) and (L3), for some $n \in \mathbb{N}$, for each $s \in S$, there is $m_s \in \mathbb{N}$ such that $w_s = m_s/n$ and $\sum_{s\in S} m_s = n$. Then, $\pi(L) = \pi\left(\sum_{s\in S} \frac{m_s}{n}\mu_s\right) = \pi[\frac{1}{n}\sum_{s\in S}(\overbrace{\mu_s + \ldots + \mu_s}^{m_s})]$. We iteratively define a collection of sets, $\{M_s\}_{s\in S}$: $M_1 = \{1, \ldots, m_1\}$, for $s \geq 2$, $M_s = \left\{\sum_{k=1}^{s-1} m_k + 1, \ldots, \sum_{k=1}^{s-1} m_k + m_s\right\}$. Moreover, let $M = \cup_{s\in S} M_s$. Also, we define a collection of assignments, $(\nu_m)_{m\in M}$ as follows: for each $m \in M$, since there is a unique $s \in S$ with $m \in M_s$, let $\nu_m = \mu_s$. Then, the lottery $\frac{1}{n}\sum_{m\in M} \nu_m$ is of equal weights and equivalent to L. $\square$

Proof of Claim 2.1. Part (1) is obvious by construction of $B^t(\cdot)$.

Part (2): Let $t \in \{0\} \cup \mathbb{N}$. Suppose $B^t(\{a\}) \cap X = \emptyset$, but $B^t(\{a\}) = B^{t+1}(\{a\})$. Let $\{i_1, \ldots, i_M\} := \{i \in I \mid \mu_S(1, i) \in B^t(\{a\})\}$, and for each $m \in \{1, \ldots, M\}$, let $a_m := \mu(1, i_m) \in B^t\{a\}$. Since $B^t(\{a\}) \cap X = \emptyset$, $a_m \notin X$, i.e., for each $m \in \{1, \ldots, M\}$, $|\mu_1^{-1}(a_m)| \geq q_{a_m}$. This inequality is strict for at least one m, as $\{a\} \in B^t(\{a\})$ and $|\mu_1^{-1}(a)| > q_a$. Thus, $\sum_{a\in\{a_1,\ldots,a_m\}} |\mu_1^{-1}(a)| = \sum_{m=1}^{M} |\mu_1^{-1}(a_m)| > \sum_{m=1}^{M} q_{a_m} \geq \sum_{a\in\{a_1,\ldots,a_m\}} q_a$, which contradits the feasibility of μ_S.

Part (3): If the claim is not true, we have $\{a\} \subsetneq B^1(\{a\}) \subsetneq \ldots \subsetneq B^t(\{a\}) \subsetneq \ldots$, which contradicts the finiteness of A. $\square$

To prove Theorems 2.1 and Part (2) of Theorem 2.3, we need the following notion and lemma.

Definition 2.5. Let $\succ \in \mathbf{P}^N$ and $P, R \in \mathcal{S}$. A t**emporary list of size m** is $(a^1, i^1, \ldots, a^m, i^m, a^{m+1})$ such that for each $\ell \in \{1, \ldots, m\}$, (1) $a^{\ell+1} \succ_{i^\ell} a^l$, (2) $p_{i^\ell, a^\ell} < r_{i^\ell, a^\ell}$, (3) $p_{i^\ell, a^{\ell+1}} > r_{i^\ell, a^{\ell+1}}$, and (4) $a^1, \ldots, a^m$ are distinct. An **improvement cycle from R to P**, denoted as $(a^1, i^1, \ldots, a^m, i^m, a^{m+1})$, is a temporary list of size m such that $a^{m+1} = a^1$.

Lemma 2.5. *Let $\succ \in \mathbf{P}^N$, $i \in N$, and $P, R \in \mathcal{S}$ be non-wasteful at $\succ$. Suppose that P stochastically dominates R at $\succ$. Then there is an improvement cycle from R to P.*

Proof. We first construct a temporary list of size 1, (a^1, i^1, a^2), where a^1 and a^2 are distinct. Since $P \neq R$, there is $i^1 \in N$ such that $P_{i^1} \neq R_{i^1}$. Thus, since P_{i^1} stochastically dominates R_{i^1} at $\succ_{i^1}$, there are $a^1, a^2 \in A$ such that $a^2 \succ_{i^1} a^1$, $p_{i^1, a^2} > r_{i^1, a^2}$, and $p_{i^1, a^1} < r_{i^1, a^1}$. Thus $a_1 \neq a_2$. Then (a^1, i^1, a^2) is the desired list.

Suppose we are given a temporary list of size m, $(a^1, i^1, \ldots, a^{m-1}, i^{m-1}, a^m)$, where $a^1, \ldots, a^m$ are distinct. Then (1) $a^m \succ_{i^{m-1}} a^{m-1}$, (2) $p_{i^{m-1}, a^{m-1}} < r_{i^{m-1}, a^{m-1}}$, and (3) $p_{i^{m-1}, a^m} > r_{i^{m-1}, a^m}$. Then, since $r_{i^{m-1}, a^{m-1}} > p_{i^{m-1}, a^{m-1}} \geq 0$, by the feasibility of P and non-wastefulness of R, we have $\sum_{j \in N} p_{j, a^m} \leq q_{a^m} = \sum_{j \in N} r_{j, a^m}$. Thus, since $p_{i^{m-1}, a^m} > r_{i^{m-1}, a^m}$, there is $i^m \in N$ such that $p_{i^m, a^m} < r_{i^m, a^m}$. Thus, since P_{i^m} stochastically dominates R_{i^m} at $\succ_{i^m}$, there is $a^{m+1} \in A$ such that $a^{m+1} \succ_{i^m} a^m$ and $p_{i^m, a^{m+1}} > r_{i^m, a^{m+1}}$. Thus $(a^1, i^1, \ldots, a^m, i^m, a^{m+1})$ is a temporary size of m. Then, if $a^{m+1} = a^\ell$ for some $\ell \in \{1, \ldots, m\}$, then the list $(a^\ell, i^\ell, \ldots, a^m, i^m, a^{m+1})$ is an improvement cycle from R to P. Othewise we continue this process. However, since $|A|$ is finite, we eventually obtain an improvement cycle from R to P. □

Proof of Theorem 2.1. Let L be a lottery with the support μ_S: ($\Rightarrow$) We show the contrapositive. Suppose that the support μ_S of L is not Pareto efficient at $\succ_S$. Then there is an $|S|-$fold replica assignment ν_S that Pareto dominates μ_S at $\succ_S$. As in Lemma 2.1, there is an equal-weight lottery $L^e = (1/|M|) \sum_{m \in M} \mu'_m$ that is equivalent to L such that for each $m \in M$ there is a unique $s(m) \in S$ with $\mu'_m = \mu_{s(m)}$. Now we define an $|S|-$fold replica assignment ν'_M: for $m \in M$, $\nu'_m = \nu_{s(m)}$. Then, ν'_M Pareto dominates μ'_M at $\succ_M$. By Lemma 2.3, the equal-weight lottery with the support ν'_M

stochastically dominates the equal-weight lottery μ'_M at $\succ$. Thus L is not sd-efficient at $\succ$.

($\Leftarrow$) We show the contrapositive. Suppose that L is wasteful (and thus not sd-efficient) at $\succ$. Let $R = \pi(L)$ be the stochastic assignment induced by L. Then there is $i \in N$, $a \in A$ with $r_{i,a} > 0$, and $b \in A$ with $b \succ_i a$ such that $\sum_{j\in N} r_{j,b} < q_b$. As $r_{i,a} > 0$, there is $s \in S$ such that $\mu_s(i_s) = a$. Then, let ν_s be an $s-$replica assignment such that $\nu_s(i_s) = b$ and for each $j \in N$, $\nu_s(j_s) = \mu_s(j_s)$. Then, the $|S|-$fold replica assignment $(\nu_s, \mu_{S\setminus\{s\}})$ Pareto dominates μ_s at $\succ_S$.

Suppose that L is non-wastful but not sd-efficient at $\succ$. Then, there is a stochastic assignment $P \neq R$ that stochastically dominates R at $\succ$. By Lemma 2.5, there is an improvement cycle, denoted by $(a^1, i^1, \ldots, a^m, i^m)$, from R to P. Then, we can find indices $s^1, \ldots, s^m \in S$ such that $\mu_{s^1}(i^1) = a^1, \ldots, \mu_{s^m}(i^m) = a^m$. Then, define an $|S|-$fold replica assignment ν_S such that $\nu_{s^1}(i^1) = a^2, \ldots, \nu_{s^{m-1}}(i^{m-1}) = a^m$, $\nu_{s^m}(i^m) = a^1$, and any other agent is assigned the same object as in μ. Then, ν_S Pareto dominates μ_S at $\succ_S$. $\square$

Proof of Theorem 2.3. We first show that TRSD^K satisfies the equal treatment of equals. Let $i, j \in N$ with $i \neq j$ and $\succ$ a problem with $\succ_i = \succ_j$. For each priority f we define another priority $f^{i\leftrightarrow j}$ to be the priority where only the positions of i and j under f are switched and the other agents have the same positions as in f. Note that the size of the support is $|\mathcal{F}(K)| \times K \times K$. Consider the lottery of the TRSD^K after $F(K) = \{f_1, \ldots, f_K\} \in \mathcal{F}(K)$ is selected. Then agents face lottery $\frac{1}{K}\sum_{g\in F(K)} TTC_{F(K)}(\succ_{F(K)}, SD_{F(K)}(\succ); g)$. Consider $F^{i\leftrightarrow j}(K) := \left\{f_1^{i\leftrightarrow j}, \ldots, f_K^{i\leftrightarrow j}\right\}$ and the lottery

$\frac{1}{K}\sum_{g\in F^{i\leftrightarrow j}(K)} TTC_{F^{i\leftrightarrow j}(K)}(\succ_{F^{i\leftrightarrow j}(K)}, SD_{F^{i\leftrightarrow j}(K)}(\succ); g)$.

Since the positions of agent i and j are just reversed, the resulting lotteries are the same except that agent i and j's stochastic assignments are switched. That is, we have

$$\frac{1}{K}\sum_{g\in F(K)} TTC_{F(K)}(\succ_{F(K)}, SD_{F(K)}(\succ); g)(i) =$$
$$= \frac{1}{K}\sum_{g\in F^{i\leftrightarrow j}(K)} TTC_{F^{i\leftrightarrow j}(K)}(\succ_{F^{i\leftrightarrow j}(K)}, SD_{F^{i\leftrightarrow j}(K)}(\succ); g)(j)$$

Now, there exist nonempty and disjoint sets $\mathcal{H}$ and $\mathcal{H}'$ such that $\mathcal{H}\cup\mathcal{H}' = \mathcal{F}(K)$ and for each $F(K) \in \mathcal{H}$, $F^{i\leftrightarrow j}(K) \in \mathcal{H}'$. Then, using the above equation and letting $\varphi(F(K), g) = \frac{1}{K}TTC[\succ_{F(K)}, SD_{F(K)}(\succ); g]$,

$$\begin{aligned}
TRSD^K(\succ)(i) &\equiv \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{F}(K)} \sum_{g\in F(K)} \varphi(F(K), g)(i) \\
&= \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{F}(K)} \sum_{g\in F^{i\leftrightarrow j}(K)} \varphi(F^{i\leftrightarrow j}(K), g)(j) \\
&= \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{H}} \sum_{g\in F^{i\leftrightarrow j}(K)} \varphi(F^{i\leftrightarrow j}(K), g)(j) + \\
&\quad + \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{H}'} \sum_{g\in F^{i\leftrightarrow j}(K)} \varphi(F^{i\leftrightarrow j}(K), g)(j) \\
&= \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{H}'} \sum_{g\in F(K)} \varphi(F(K), g)(j) + \\
&\quad + \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{H}} \sum_{g\in F(K)} \varphi(F(K), g)(j) \\
&= \frac{1}{\binom{n!}{K}} \sum_{F(K)\in\mathcal{F}(K)} \sum_{g\in F(K)} \varphi(F(K), g)(j) = TRSD^K(\succ)(k).
\end{aligned}$$

The equality of the first term in the second and third line comes from the following: $[F(K) \in \mathcal{H}$ and $g \in F^{i\leftrightarrow j}(K)] \Leftrightarrow [F^{i\leftrightarrow j}(K) \in \mathcal{H}'$ and $g \in F^{i\leftrightarrow j}(K)] \Leftrightarrow [F'(K) \in \mathcal{H}'$ and $h \in \mathcal{H}']$. Similarly, the equality of the second term in the second and third line comes from the following: $[F(K) \in \mathcal{H}'$ and $g \in F^{i\leftrightarrow j}(K)] \Leftrightarrow [F^{i\leftrightarrow j}(K) \in \mathcal{H}$ and $g \in F^{i\leftrightarrow j}(K)] \Leftrightarrow [F'(K) \in \mathcal{H}$ and $h \in F'(K)]$. Hence, the TRSD^K satisfies the equal treatment of equals.

Part (1): Note that to show that $TRSD^K$ stochastically dominates RSD, we need to show that for some problem $\succ$, $TRSD^K \neq RSD$ due to the weakly stochastic dominance just proved above. If $|N| \leq 3$, then RSD is sd-efficient (BM), and thus $TRSD^K = RSD$. Suppose $|N| \geq 4$. Example 2.3 shows that for $|N| = 4$, $TRSD^K \neq RSD$. The extension to the case of $|N| \geq 5$ is straightforward.

Part (2): Suppose RSD is not sd-efficient for some N, A, and q. Then there is a problem $\succ$ such that $RSD(\succ)$ is not sd-efficient at $\succ$. Let $R := RSD(\succ)$. We first show

Claim 2.2. there exist $\bar{K} \leq n$ and $F(\bar{K}) := \{f_1, \ldots, f_{\bar{K}}\}$ such that $SD_{F(\bar{K})}(\succ)$ is not Pareto efficient in the $|K|$-fold replica problem. First consider the case where R is wasteful at $\succ$. Then there is $i \in N$, $a \in A$ with $r_{i,a} > 0$, and $b \in A$ with $b \succ_i a$ such that $\sum_{j \in N} r_{j,b} < q_b$. Then there is $f_1, f_2 \in F$ such that $SD_{f_1}(\succ)(i) = a$ and $\sum_{j \in N} |SD_{f_2}(\succ)(j)| < q_b$. Take $\bar{K} = 2$ and $F(\bar{K}) := \{f_1, f_2\}$. Then $SD_{F(\bar{K})}(\succ)$ is not Pareto efficient at $\succ$. Consider another case where R is non-wasteful but not sd-efficient at $\succ$. Then there is a stochastic assignment P such that P stochastically dominates R at $\succ$. Then, by Lemma 2.5, there is an improvement cycle $(a^1, i^1, a^2, i^2, \ldots, a^m, i^m, a^{m+1})$ from R to P. Let $\bar{K} := m$. Then, since $a^1, \ldots, a^m$ are distinct, we have $\bar{K} \leq |A|$. Moreover, since $r_{i^\ell, a^\ell} > 0$ for each $\ell \in \{1, \ldots, m\}$, there is $F(\bar{K}) := \{f_1, \ldots, f_{\bar{K}}\}$, where $F(\bar{K})$ allows for duplicate elements, such that for each $\ell \in \{1, \ldots, m\}$, $SD_{f_k}(\succ)(i^\ell) = a^\ell$. Then an assignment ν where for each $i \in \{1, \ldots, m\}$, $\nu(i^\ell) = a^{\ell+1}$, Pareto dominates $SD_{F(\bar{K})}(\succ)$. Hence $SD_{F(\bar{K})}$ is not Pareto efficient at $\succ$. Thus the proof of Claim 2 is completed.

Let $K \geq \bar{K}$. Then there is $F(K) \subseteq F$ such that $F(\bar{K}) \subseteq F(K)$. Then, by Claim 2.2, $SD_{F(K)}(\succ)$ is not Pareto efficient. Thus, since $TRSD_{F(K)}(\succ_{F(K)}, SD_{F(K)}; g)$ for some $g \in F$ is Pareto efficient, we have $TRSD_{F(K)}(\succ_{F(K)}, SD_{F(K)}; g) \neq SD_{F(K)}(\succ)$. Therefore $TRSD^K \neq SD$. □

Chapter 3

Fairness and Efficiency in Strategy-proof Object Allocation Mechanisms

This chapter is based on Nesterov (2014).

3.1 Introduction

The optimal allocation of goods among individuals is one of the core issues in economics. Normally, researchers analyze this issue using the well-established concepts of markets and auctions, in which individuals receive goods in exchange for transfers. However, in a variety of real-life situations, these transfers are not available for either ethical, institutional or other reasons. Recent literature analyzes numerous examples of such situations. These range from student assignment to primary schools (Abdulkadiroğlu and Sönmez, 2003) and job placement for graduates (Roth, 1984), Coles et al., 2010), to on-campus housing (Chen and Sönmez, 2002), organ donation (Roth, Sönmez and Ünver, 2005) and distributing military supplies (Kesten and Yazici, 2012).

In this paper we study the simplest version of this class of problems: the assignment problem,[1] where a set of indivisible objects is allocated to a set of agents solely according to their preferences and such that each agent receives precisely one object. The assignment problems have two stages: first agents report their (ordinal) preferences over objects and then, based on these preferences and using some systematic procedure which we call *a mechanism*, the final assignment is determined. Given the reported preferences and the assignment, we can judge whether the mechanism is efficient (the assignment is not dominated in a certain sense), fair (the agents are treated fairly according to certain criteria), and incentive compatible (agents weakly prefer to report their preferences truthfully). The mutual compatibility of these three types of properties is the focus of this paper.

We can approach the properties of a mechanism using two types of solutions to an assignment problem: either a final deterministic solution or, more generally, an intermediate stochastic solution. The deterministic solution is a one-to-one correspondence between the set of agents and the set of objects; this correspondence is called *matching*. A matching precisely prescribes who gets what. However, deterministic solutions are very unfair (ex-post) and in order to restore some fairness at least ex-ante, we often use randomization and focus on the intermediate stochastic solution. In contrast to matching, the stochastic solution is a matrix of assignment probabilities such that *in expectation* each agent receives precisely one object and each object is assigned to precisely one agent. This probability matrix is called *a random assignment*.

Since the formal introduction of the assignment problem by Hylland and Zeckhauser (1979) there has been a search for "nice" mechanisms that would satisfy these major properties: incentive compatibility, efficiency, and fairness. Hylland and Zeckhauser (1979) themselves propose a pseudo-market mechanism that optimally satisfies the latter two properties: the assignments are always ex-ante Pareto efficient (the random assignments are never Pareto dominated) and envy-free (all agents prefer their individual random assignments to the random assignments of others). However, the reliability of these

[1] The assignment problem is also known as the one-sided matching problem and the house allocation problem.

efficiency and fairness properties is doubtful because the mechanism is not incentive compatible. Since the utilities are private information, the agents can strategically misreport them in order to get a better random assignment. Therefore, the mechanism is not necessarily ex-ante efficient and envy-free under the *true* preferences of the agents.

The further search for a reliably "nice" mechanism gave rise to a series of negative results. Gale (1987) was the first to conjecture that for an assignment problem with at least three agents, no mechanism can satisfy ex-ante Pareto efficiency, strategy-proofness, and *anonymity.* (Anonymity is a weak notion of fairness that requires that any two agents with identical reported utilities get the same individual random assignment.) Later, Zhou (1990) showed a stronger result, where instead of anonymity he used *symmetry.* (Symmetry is implied by anonymity, it requires that any two agents with identical reported utilities get the same *expected utility* and not necessarily the same random assignments as in the case of anonymity).

Subsequently, in their seminal paper Bogomolnaia and Moulin (2001), hereinafter referred to as BM, show a similar but logically independent impossibility result. BM consider agents with strict *ordinal* preferences over objects (as opposed to utilities in the papers mentioned above). The preferences of each agent therefore create something that can be seen as a partial order over the individual random assignments: the agent always prefers one random assignment over another if it first-order stochastically dominates the other assignment. Based on this partial order, BM redefine the efficiency concept: they call a random assignment *ordinally efficient* if it is not stochastically dominated by any other random assignment.[2] Using this criterion, they show the following impossibility result: for the assignment problem with at least four agents, no mechanism can satisfy strategy-proofness, ordinal efficiency, and *equal treatment of equals.* (The latter is a weak fairness criterion that requires that agents with the same ordinal preferences get the same random assignments.)[3]

[2]Ordinal efficiency is also often referred to as *sd-efficiency*; it is implied by ex-ante efficiency.

[3]Equal treatment of equals implies anonymity but is logically independent from sym-

The goal of this paper is to further study the feasibility set of the "nice" mechanisms. We do so by revealing the tradeoff between fairness and efficiency in assignment problems. To make our point, throughout the paper we consider only strategy-proof mechanisms and change the combination of efficiency and fairness criteria. In choosing these criteria we adhere to the ones that we think are most relevant for the real-life applications.

The most practically relevant efficiency criterion is *ex-post efficiency*, which is widely used in the random assignment literature as well as in the matching literature. By definition, the ex-post efficient mechanism exclusively induces the random assignments that can be expressed as lotteries over Pareto efficient matchings. In other words, ex-post efficient mechanisms are always able to induce a Pareto efficient *deterministic* outcome, or matching. However, only ordinal efficiency guarantees that *any possible* deterministic outcome of a random assignment is Pareto efficient. Therefore, ex-post efficiency is implied by ordinal efficiency, which in turn is implied by ex-ante efficiency.[4] Since in the real-life applications we almost exclusively deal with deterministic assignments, ex-post efficiency is a reasonable minimum efficiency requirement for a mechanism.

Regarding fairness, throughout the paper we use envy-freeness and a few weaker concepts related to it. Introduced by Foley (1967), envy-freeness "quickly became the dominant argument of justice within microeconomic theory" (Moulin , 1995).[5] The other, weaker concepts that we use share the following requirement with envy-freeness: they all require that for a (certain) subset of agents each agent must prefer (in a certain way) her own assignment to the assignments of other agents within this subset. In other words, agents in this subset should not envy each other in a specific way. Depending on how strict the fairness notion is, the size of this subset varies as does the strictness of the envy among agents in this subset. We introduce these fairness concepts

metry, since the latter does not require equal random assignments for equals unlike the other two notions.

[4]For the case of $N = 3$ agents ex-post efficiency coincides with ordinal efficiency; for the case of $N = 2$ agents all three efficiency criteria are equivalent.

[5]An extensive survey of results on envy-freeness and adjacent concepts can be found in Arnsperger (1994), as well as Moulin (1995) and its newer edition from 2014.

together with the related results below.

There are five main results in this paper: three impossibilities and two characterizations.

The first result states a general impossibility regarding ex-post efficiency. We show that when there are at least three agents, there is no ex-post efficient, envy-free, and strategy-proof mechanism. In fact, Lemma 3.1 shows an even stronger result in which envy-freeness and strategy-proofness are substituted by a pair of weaker properties.

This result is most relevant for deterministic assignment mechanisms, such as *dictatorship* mechanisms[6] or mechanisms based on the *top trading cycle* algorithm (TTC).[7] These deterministic mechanisms are usually required to be Pareto efficient and strategy-proof, but they can be very unfair ex-post.[8] That is why modifications of these mechanisms may involve randomization in order to restore fairness ex-ante. However, as implied by the first result, in these modifications envy-freeness can only be achieved at the cost of either ex-post efficiency or strategy-proofness.

In the rest of the paper, we further study the tradeoff between fairness and efficiency by relaxing the envy-freeness requirement and using a few weaker fairness criteria instead.

The second result deals with a direct generalization of envy-freeness: *weak envy-freeness*. Weak envy-free random assignment eliminates only inevitable envy, i.e., envy that one agent has for another's assignment for *any* cardinal utilities that are consistent with her ordinal preferences. We also call this type of envy strong envy. In contrast, normal envy-freeness eliminates any possible envy, i.e., envy that one agent might have for another agent's assignment for at least some cardinal utilities that are consistent with her ordinal preferences.

[6]In a dictatorship mechanism the assignment is determined by one of the agents, though the acting agent may be constantly changing.

[7]The TTC mechanism is attributed to David Gale, it was introduced in Shapley and Scarf (1974). The mechanisms based on TTC are used in various settings: in school choice, organ donation, and housing problems; for details see Abdulkadiroğlu and Sönmez (2003), Roth et al. (2004) and Abdulkadiroğlu and Sönmez (2010) correspondingly.

[8]In fact, ex-post fairness is an extremely restrictive property, as shown by Kesten and Yazici (2012).

Formally, a random assignment is weak envy-free if, for each agent, her own assignment is not strictly stochastically dominated by any other agent's assignment. As an example of weak envy-freeness, consider the following random assignment of three houses h_1, h_2, h_3 to three agents a_1, a_2, a_3:

$$\begin{array}{c|ccc} & h_1 & h_2 & h_3 \\ \hline a_1 & .5 & .0 & .5 \\ a_2 & .3 & .4 & .3 \\ a_3 & .2 & .6 & .2 \end{array},$$

where agent a_1 receives house h_1 with the probability .5, house h_2 with zero probability and so forth. Let the preferences be such that all three agents a_1, a_2, a_3 prefer house h_1 to house h_2, and house h_2 to house h_3. This random assignment is weakly envy-free because none of the individual lotteries stochastically dominates another. For an opposite example, assume that all agents prefer house h_1. Then the assignment is no longer weak envy-free, since agent a_1 strongly envies the other two agents because he gets lower chances to get the best house and higher chances to get the worst house.

The second result states that for $N \geq 4$, there is no weak envy-free, ordinally efficient, and strategy-proof mechanism. Together with the previous impossibility result, it shows the tradeoff between efficiency and fairness in terms of envy. Precisely, given strategy-proofness, when relaxing the fairness criterion from envy-freeness to weak envy-freeness, the feasibility threshold in terms of efficiency shifts from ex-post efficiency to ordinal efficiency.

This result is very close to the impossibility result in BM, namely the mutual incompatibility of strategy-proofness, ordinal efficiency, and equal treatment of equals. Equal treatment of equals and weak envy-freeness are logically independent, though the latter is arguably more relevant in practice for two main reasons. First, weak envy-freeness applies to the full set of preference profiles, while equal treatment of equals restricts assignments of agents with identical preferences. Second, equal treatment of equals can be seen as just envy-freeness for equals. Indeed, agents with identical preferences will never envy each other if and only if they receive identical assignments. But these agents do not necessarily envy each other in the first place, and thus they can be satisfied with the assignment even if equal treatment does not hold. Hence, if we assume that our fairness concerns come from the *possible* envy among agents, as opposed to *inevitable* envy, then equal treatment of equals might excessively restrict the assignments of the rare agents with identical preferences but in the same time it completely disregards all other agents. Therefore, an average weak envy-free assignment is (roughly speaking) more satisfactory for agents than an assignment that satisfies equal treatment of equals.[9]

The third result of the paper is the characterization of the *random serial dictatorship mechanism* (RSD) for the case of three agents. RSD is perhaps the most popular assignment mechanism in practical applications. RSD proceeds as follows: the agents are ordered randomly and then, according to this order, each of them picks her most preferred object among the remaining objects. RSD is strategy-proof and ex-post efficient, but it is not always ordinally efficient, as shown by BM. For the case with three agents, BM also characterize RSD as the unique ex-post efficient, strategy-proof mechanism that satisfies equal treatment of equals. We strengthen this result by showing that RSD is the unique strategy-proof and ex-post efficient mechanism that eliminates strong envy between agents with identical preferences, the property that we call *weak envy-freeness among equals.*

Weak envy-freeness among equals can be seen as a natural relaxation of either the equal treatment of equals or the weak envy-freeness. On the one hand, weak envy-freeness among equals does not restrict the individual random assignments for two agents with identical preferences *unless* one of them strictly envies another (which might be redundant as explained above). On the other hand, weak envy-freeness among equals does not restrict the individual random assignments of two agents if one of them strictly envies the other *unless* they have identical preferences, while weak envy-freeness does so as if the mechanism designer must guarantee equitable treatment even for different agents (which may also an excessive requirement).

This result implies the characterization by Bogomolnaia and Moulin (2001) since weak envy-freeness among equals follows from equal treatment of equals.

[9] We further discuss the relevance of weak envy-freeness in section 5.

Another implication is that for $N = 3$ RSD can be characterized as the unique mechanism that is strategy-proof, ex-post efficient, and weak envy-free (for all agents). These results underline the central role of RSD among the strategy-proof and ex-post efficient mechanisms and fit nicely into the series of other characterization and equivalence results regarding RSD.[10]

The fourth result is another characterization of RSD in case $N = 3$; it is strongly related to the previous result, though it is logically independent. This time, RSD is characterized as a unique mechanism that is ex-post efficient, strategy-proof, and that satisfies symmetry, the fairness notion used by Zhou (1990). Similarly to weak envy-freeness among equals, symmetry requires that agents with identical cardinal preferences receive assignments that do not cardinally dominate one another, i.e., deliver the same utility. Since RSD is not ex-ante efficient, this characterization implies the impossibility by Zhou (1990). Even though this characterization is logically independent from the previous result, it also implies the characterization in BM (since symmetry is implied by equal treatment of equals).

In the last part of the paper we focus on an alternative approach to fairness: the so-called "fare share guaranteed." Here, the agents' assignments are compared not to one another's, as in envy-freeness, but to the "fair" assignment of equal division such that each agent receives each object with equal probability $\frac{1}{N}$. And if the assignment ordinally dominates the equal division then it is considered to be fair and to satisfy *equal division lower bound*.[11]

In the fifth result of the paper we show that for $N \geq 4$ there is no strategy-proof and ordinally efficient mechanism that satisfies equal division lower bound.

This result is important for a large class of mechanisms that satisfy equal division lower bound by construction. In these mechanisms, agents always

[10] Knuth (1996) and Abdulkadiroğlu and Sönmez (1998) show the equivalence between the symmetrized TTC mechanism and RSD. This result is further generalized in Bade (2014) to the set of all symmetrized Pareto optimal, strategy-proof and non-bossy mechanisms.

[11] An extensive review on comparison to equal division and other notions of fairness for allocation rules can be found in Moulin (2014) and Thomson (2007).

have the opportunity to get at least the equal division assignment. For example, in the pseudo-market mechanism proposed in Hylland and Zeckhauser (1979) the agents have equal budgets with which they purchase probability shares of objects at competitive equilibrium prices. As a result, in any feasible random assignment, the budget is sufficient to purchase the assignment that is at least as good as the equal division. Therefore, such a mechanism inevitably lacks either ordinal efficiency or strategy-proofness (in fact, the latter is the case since the mechanism is ex-ante efficient, which implies ordinal efficiency).

Another example of a mechanism that satisfies equal division lower bound by construction would be a mechanism that endows agents with the equal division assignment and then allows them to exchange the probability shares voluntarily. Abdulkadiroğlu and Sönmez (1998), for instance, endow the agents with some random object, which the agents can then exchange according to TTC. Since the endowment is random and the exchange is voluntary, the mechanism inevitably satisfies equal division lower bound. Similarly, in a mechanism proposed by Kesten (2009), agents are directly endowed with the equal division assignment. They can then exchange the probability shares according to TTC (with a restriction on the size of the cycles). The impossibility result implies that any of these or the analogous mechanisms are either not strategy-proof or not ordinally efficient. Indeed, Abdulkadiroğlu and Sönmez (1998) show that their mechanism is equivalent to RSD, which is strategy-proof but not ordinally efficient; and Kesten (2009) shows that his mechanism is equivalent to the probabilistic serial mechanism (first introduced by BM), which is ordinally efficient but not strategy-proof.

Despite the negative results presented in this paper we can, however, still hope to find a strategy-proof, fair, and efficient mechanism in some relevant cases. For large markets in which every object has an increasing number of copies (for example, one can think of slots in one school as copies of a unique slot; the number of slots increases while the number of schools remains the same), Che and Kojima (2010) show that RSD is asymptotically ordinally efficient. For a similar large market, Kojima and Manea (2010) show that the

Table 3.1: Summary of results

		Strategy-proof mechanisms			
		Envy-free	Weak envy-free	Equal division lower bound	Equal treatment of equals
Ex-post efficient	$N = 3$	$\emptyset$ (Theorem 3.1; BM*)	RSD! (Corollary 3.2)	RSD	RSD! (Corollary 3.1; BM)
	$N > 3$	$\emptyset$ (Theorem 3.1)	RSD (BM)	RSD	RSD
Ordinally efficient	$N > 3$	$\emptyset$	$\emptyset$ (Theorem 3.2)	$\emptyset$ (Theorem 3.3)	$\emptyset$ (BM)

Notes: The table presents the mutual compatibility of fairness and efficiency within the set of strategy-proof random assignment mechanisms. $\emptyset$ denotes the empty set, exclamation mark denotes uniqueness, BM stands for Bogomolnaia and Moulin (2001).
*The case of three agents is also mentioned by BM, p. 310, though informally.

probabilistic serial mechanism is asymptotically strategy-proof.[12] Therefore, the impossibility results presented here do not hold asymptotically for these types of large markets.

It should also be mentioned that some of the results of this paper are limited by the nature of the standard framework that we use. In a more general setting where the number of houses may be higher than the number of agents (especially in the case with a null object), the agents have a richer strategy set and thus one cannot directly transfer the results to that setting. For instance, in such settings RSD is no longer ex-post efficient for some preference profiles; it can also be dominated by another strategy-proof mechanism (see Erdil, 2014, for these and other results in the general setting). However, the negative results must hold, since the standard setting is a special case of the general setting.

Table 3.1 summarizes the main findings of this paper as well as the relevant results of BM.

[12]Based on the probabilistic serial mechanism Budish et al. (2013) develop fair and efficient mechanisms for various non-standard settings.

The paper proceeds as follows: Section 2 introduces the framework, section 3 presents the first impossibility (Theorem 3.1), section 4 covers the two characterizations (Proposition 3.1 and Proposition 3.2). These results are then used in section 5, which presents the second impossibility (Theorem 3.2). Section 6 examines the third impossibility (Theorem 3.3). Section 7 concludes by discussing the implications of the findings and the remaining open questions.

3.2 The Model

In this section we introduce the framework: we define the assignment problem, the random assignment mechanism, and its properties.

Let $A = \{a_1, a_2, ..., a_N\}$ be the set of N agents and $H = \{h_1, h_2, ..., h_N\}$ be the set of N houses. Each agent $a \in A$ is endowed with a strict preference relation $\succ_a$ on H with a corresponding weak preference relation $\succcurlyeq_a$. A set of individual preferences of all agents constitutes a **preference profile** $\succ = (\succ_a)_{a \in A}$. Let $\mathcal{R}$ be the set of all possible individual preferences and $\mathcal{R}^N$ be the set of all possible preference profiles. In what follows we assume that the sets A and H are fixed and that the house allocation problem is defined by the preference profile $\succ$ only.

Each assignment problem has either a deterministic solution, called matching, or a probabilistic solution, called random assignment. A **random assignment** P is a doubly stochastic matrix of size N. Each element $P_{a,h}$ of the matrix P represents a probability of agent a being assigned house h. Let $\mathcal{P}$ be a set of all possible random assignments P.

A **matching** μ is a random assignment whose elements can only be zeros or ones, so that μ precisely prescribes which agent receives which house. Let $\mathcal{M}$ be a set of all possible matchings μ. According to the Birkhoff-von Neumann theorem, any random assignment P can be represented as a lottery over the set of matchings $\mathcal{M}$ (but this representation is not necessarily unique). For this reason, and since agents care only about their own assignment, we can concentrate on random assignments without specifying the exact matchings that these random assignments correspond to.

In order to be able to compare different random assignments we need the following definitions. A set of houses that agent a weakly prefers to some house h is the **upper contour set of house h at $\succ_a$**: $U(\succ_a, h) = \{h' \in H : h' \succcurlyeq_a h\}$. For example, the upper contour set of the most preferred house is always this same house, the upper counter set of the second most preferred house $-$ the two best houses, and so forth.

Given the individual random assignment P_a, the overall probability of agent a being assigned some house that is at least as good as house h is her **surplus at h under P_a**: $F(\succ_a, h, P_a) = \sum_{h' \in U(\succ_a, h)} P_{a,h'}$. In other words, the surplus at h is the probability of being assigned some object from the upper contour set of h.

An individual random assignment P_a **ordinally dominates** another individual random assignment P'_a **at $\succ_a$** (denoted by $P_a \geq_a P'_a$) if it first-order stochastically dominates it. The equivalent condition is that all surpluses of P_a weakly exceed the surpluses of P'_a: for each $h \in H$ $F(\succ_a, h, P_a) \geq F(\succ_a, h, P'_a)$. A **strict ordinal domination** (denoted by $P_a >_a P'_a$) occurs under the additional condition that the two random assignments are not identical. Finally, a random assignment P is said to **dominate** another random assignment P' if it dominates for all agents simultaneously; P **strictly dominates** P' if the assignments are not identical.

3.2.1 Properties of a mechanism

From here on we deal with systematic procedures called *mechanisms* that associate each preference profile $\succ \in \mathcal{R}^N$ with some random assignment $P \in \mathcal{P}$: $P = \varphi(\succ)$, where φ denotes a mechanism.

Efficiency. For a matching there is a single definition of efficiency: a matching is **(Pareto) efficient at some preference profile** if it is not dominated by any other matching at this preference profile. For a random assignment, on the other hand, efficiency can be defined in three ways: ex-post, ordinal, and ex-ante (the latter we define at the end of this section). A random assignment is **ex-post efficient (ExPE) at a preference profile** if it can be represented as a lottery over efficient matchings. A random assignment is **ordinally efficient (OE) at a preference profile** if it is not stochastically dominated by any other random assignments at this preference profile. A mechanism is said to be **ex-post efficient (ordinal efficient)** if for any preference profile it results in an ex-post efficient (ordinally efficient) random assignment.

Strategy-proofness. A mechanism φ is **strategy-proof (SP)** if at any preference profile no agent can benefit by misreporting her preferences: for each $a \in A$, for each $\succ \in \mathcal{R}^N$ and for each $\succ'_a \in \mathcal{R}$ the following holds: $\varphi(\succ) \geq_a \varphi_a(\succ'_a, \succ_{-a})$. In other words, under a strategy-proof mechanism, truth-telling is always a dominant strategy for every agent.

Now we introduce an auxiliary notion of incentive compatibility which is weaker than strategy-proofness; we use this property for the first impossibility result below. This notion restricts the set of (potentially) profitable strategies for agents. A mechanism is **upper shuffle-proof (USP)** if no agent a can change her surplus at some object h by "shuffling" the objects that are *strictly* better than h (or misreporting the preferences within the upper contour set of h excluding h itself). Formally, for each $a \in A, h \in H$, and for each $\succ \in \mathcal{R}^N, \succ'_a \in \mathcal{R}$ such that $U(\succ_a, h) = U(\succ'_a, h)$, the following holds: $F(\succ_a, h, \varphi_a(\succ)) - \varphi_{ah}(\succ) = F(\succ_a, h, \varphi_a(\succ')) - \varphi_{ah}(\succ')$ (the difference represents the sum of assignment probabilities for houses that are strictly better than h). For example, if $N = 3$ upper shuffle-proofness requires that no agent can benefit—in terms of the sum of assignment probabilities for the top two houses—by swapping these two houses. The agents could still possibly benefit: either in some other respect (not in terms of the surplus of the second best object), or from using other strategies (that involve other swaps).[13]

Fairness. A random assignment P is **envy-free (EF)** if every agent prefers her assignment to any other agent's assignment: for each $a, a' \in A$ $P_a \geq_a P_{a'}$. A random assignment P is **weak envy-free (wEF)** if no agent strictly prefers some other agent's assignment: there do not exist $a, a' \in A$ such that $P_{a'} >_a P_a$. Another widely used notion of fairness is the **equal treatment of equals (ETE)**: for each $a, a' \in A$ with $\succ_a = \succ_{a'}$ the individual random assignments are identical: $P_a = P_{a'}$. A weaker combination

[13] Upper-shuffle-proofness is the same as lower invariance in Mennle and Seuken (2014).

of the previous two properties is called **weak envy-freeness among equals**. A random assignment P is weak envy-free among equals if for any two agents a, a' with identical preferences $\succ_a = \succ_{a'}$ none of them strictly prefers the assignment of the other: $P_{a'} \not\succ_a P_a$.[14]

Another approach to fairness is the so-called "fair-share guaranteed." It requires that each agent weakly prefers her individual assignment to the fair division assignment. Formally, P satisfies the **equal division lower bound (EDLB)** if $P \geq ED$, where ED denotes the equal division random assignment.

Next, we introduce two auxiliary notions of fairness: upper envy-freeness and the strong equal treatment of equals. One of them is a modification of envy-freeness: it restricts the set of agents "eligible to envy" to only those agents that have the same upper contour set of some house, and the envy is considered only for this particular house. Formally, a random assignment P is **upper envy-free (UEF)** if any two agents with identical upper contour sets of some house h receive equal assignment probabilities of h: for each $a, a' \in A, h \in H$ such that $U(\succ_a, h) = U(\succ_{a'}, h)$ it follows that $P_{ah} = P_{a'h}$.

The other fairness notion is a generalization of equal treatment of equals. A random assignment P satisfies the **strong equal treatment of equals (SETE**) if any two agents with identical preferences from the top house down to some particular house receive identical assignments from the top down to that house. Observe that upper envy-freeness and strong equal treatment of equals differ from the definitions of envy-freeness and equal treatment of equals in that the set of agents that can be compared is different, namely, is restricted for envy-freeness and enlarged for the equal treatment of equals.

We also introduce two fairness notions and one efficiency notion for the cardinal framework.[15] Assume that each agent $a \in A$ reports her utility

[14] One can see how these properties are related using the following logic. Equal treatment of equals can be seen as (strong) envy-freeness among equals: if two agents have the same preferences, one of them never envies the other if and only if they have identical random assignment. Therefore weak envy-freeness among equals is the weak form of this property (of envy-freeness among equals), similar to the relationship between envy-freeness and weak envy-freeness.

[15] We only need the cardinal framework for the second characterization, apart from that we use the ordinal framework.

$u_a = \{u_{ah}\}_{h \in H} : u_{ah} \in \mathbb{R}$ for each object $h \in H$. A random assignment P is **symmetric** if every two agents a and a' with the same reported utilities $u_a = u_{a'}$ receive equal expected utility: $\sum u_{ah} P_{ah} = \sum u_{a'h} P_{a'h}$. A random assignment P is **anonymous** if the same two agents in addition receive identical random assignments: $P_a = P_{a'}$. A random assignment P is **ex-ante efficient at utility** $U = \{u_a\}_{a \in A}$ if there does not exist any other random assignment P' such that for each agent a assignment P provides an (expected) utility that is at least as high as the utility provided by the assignment P': $\sum P_{ah} u_{ah} \geq \sum P'_{ah} u_{ah}$, and, at least for one of the agents, the inequality is strict.

Finally, a mechanism is said to satisfy one of the fairness or efficiency properties introduced above if it always induces random assignments with this property.

The efficiency notions can be logically ordered: ex-post efficiency is implied by ordinal efficiency, which in turn is implied by ex-ante efficiency.

The fairness notions can be logically ordered as well, as the following remark shows. Figure 3.2.1 illustrates the remark.

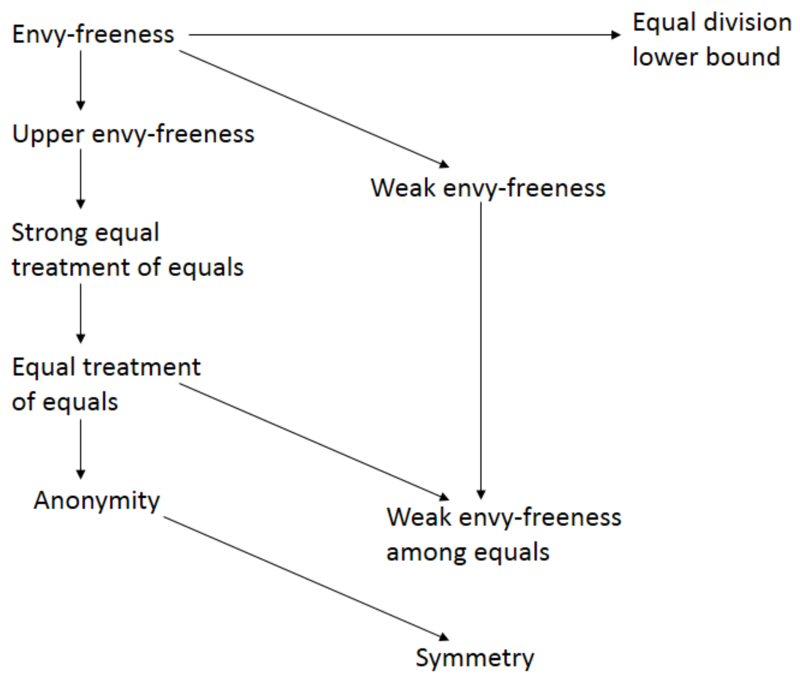
Remark. The following logical relations hold:

1. envy-freeness $\Longrightarrow$ upper envy-freeness $\Longrightarrow$ strong equal treatment of equals $\Longrightarrow$ equal treatment of equals $\Longrightarrow$anonymity$\Longrightarrow$symmetry;
2. envy-freeness $\Longrightarrow$ weak envy-freeness;
3. envy-freeness $\Longrightarrow$ equal division lower bound;
4. weak envy-freeness, equal division lower bound and upper envy-freeness (as well as strong equal treatment of equals and equal treatment of equals) are logically independent.

The proof of these relations can be found in the appendix.

We have now prepared all the necessary definitions and their logical relations to study the first impossibility result presented in the next section.

Figure 3.2.1: Logical relations between fairness notions.



Notes: Arrows denote logical implications for different fairness notions.

3.3 First Impossibility Result

We begin by studying the tradeoff between the properties of a mechanism when fairness is of a higher concern than efficiency. The following theorem considers the set of strategy-proof mechanisms that are moderately efficient (at least ex-post efficient) and very fair (envy-free, which implies all other fairness criteria). The set of such mechanisms turns out to be empty:

Theorem 3.1. *For $N \geq 3$ there does not exist a mechanism that is ex-post efficient, strategy-proof, and envy-free.*

The result above is a direct corollary to a stronger result of Lemma 3.1:

Lemma 3.1. *There does not exist a mechanism that is ex-post efficient, upper-shuffle-proof, and upper-envy-free.*

Proof. We first prove the claim for $N = 3$ and we do it by contradiction. Suppose there exists a mechanism φ satisfying ex-post efficiency, upper shuffle-proofness an upper envy-freeness.

For convenience of the proofs we use a novel notation: instead of a preference profile we use a rank table, that is, a matrix $N \times N$ with rows (columns) corresponding to agents (houses). The elements of the rable are the ranks of the respective house in the preferences of a respective agent. For instance, for the preference profile $\succ$:

$$\succ\colon \begin{array}{c||ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_2 & h_1 & h_3 \end{array},$$

the corresponding rank table $r(\succ)$ is as follows (the superscripts denote the probabilities in the random assignment $\varphi(\succ)$):

$$r(\succ) = \begin{array}{ccc} 1^{\frac{1}{2}} & 2^{\frac{1}{4}} & 3^{\frac{1}{4}} \\ 1^{\frac{1}{2}} & 3^{0} & 2^{\frac{1}{2}} \\ 2^{0} & 1^{\frac{3}{4}} & 3^{\frac{1}{4}} \end{array}.$$

To see that $\varphi(\succ)$ is indeed as shown let us begin with the assignment probabilities of house h_1. Agent a_3 receives zero probability $\varphi_{a_3h_1}(\succ) = 0$ due to the ExPE of φ. Agents a_1 and a_2 receive equal probabilities $\varphi_{a_1h_1}(\succ) =$

$\varphi_{a_2h_1}(\succ) = \frac{1}{2}$ since φ satisfies SETE (implied by UEF), otherwise the agent who received less of her top house h_1 might have envied another agent. Next, consider the assignment probabilities of house h_2. Since agent a_2 dislikes house h_2 while agent a_3 prefers this house over others, agent a_2 is never assigned h_2 due to the ExPE of φ: $\varphi_{a_2h_2}(\succ) = 0$. Therefore agent a_2 is left with a one half of probability of house h_3: $\varphi_{a_2h_3}(\succ) = 1 - \varphi_{a_2h_1}(\succ) = \frac{1}{2}$. Finally, notice that the remaining assignment probability of house h_3 is spread equally between agents a_1 and a_3 due to the UEF of φ (their upper contour sets at h_3 are identical). Thus, $\varphi_{a_1h_3}(\succ) = \varphi_{a_3h_3}(\succ) = \frac{1}{4}$ and $\varphi_{a_1h_2}(\succ) = \frac{1}{4}, \varphi_{a_3h_2}(\succ) = \frac{3}{4}$.

Next consider another preference profile $\succ'$ that differs from $\succ$ in that agent a_3 copies the report of agent a_1:

$$r(\succ') = \begin{matrix} 1^{\frac{1}{3}} & 2^{\frac{1}{2}} & 3^{\frac{1}{6}} \\ 1^{\frac{1}{3}} & 3^{0} & 2^{\frac{2}{3}} \\ 1^{\frac{1}{3}} & 2^{\frac{1}{2}} & 3^{\frac{1}{6}} \end{matrix} .$$

Regarding house h_1 in this random assignment agents receive equal probability shares $\varphi_{a_1h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \varphi_{a_2h_1}(\succ') = \frac{1}{3}$ since φ is SETE. Next, due to the ex-post efficiency of φ agent a_2 receives zero probability of being assigned house h_2 as before: $\varphi_{a_2h_2}(\succ') = 0$. Therefore we conclude that $\varphi_{a_2h_3}(\succ') = \frac{2}{3}$ and, again using SETE, $\varphi_{a_1h_2}(\succ') = \varphi_{a_3h_2}(\succ') = \frac{1}{2}$ and $\varphi_{a_1h_3}(\succ') = \varphi_{a_3h_3}(\succ') = \frac{1}{6}$.

Note that φ cannot satisfy USP since when shifting from $\succ_{a_3}$ to $\succ'_{a_3}$ the agent's a_3 upper contour set at h_3 remains the same ($U(\succ_{a_3}, h_3) = U(\succ'_{a_3}, h_3)$) but the assignment probability has changed. This contradiction completes the proof for $N = 3$.

For $N > 3$ consider the following preference profile $\succ'' \in \mathcal{R}^N$. Agents with indices higher than 3 prefer a house with a corresponding index to all others: $\forall a_i \in A : i > 3, \forall h \in H : h \neq h_i \implies \succ''_{a_i}: h_i \succ''_{a_i} h$. Additionally let the first three agents prefer the first three houses to any other house: $\forall a_i \in A : i, j = 1, 2, 3, \forall h \in H : h \neq h_j \implies \succ''_{a_i}: h_j \succ''_{a_i} h$. Their preferences for the first three houses are as follows:

$$r(\succ''):\begin{array}{cccccc} 1 & 2 & 3 & ... & N-1 & N \\ 1 & 3 & 2 & ... & N-1 & N \\ 1 & 2 & 3 & ... & N-1 & N \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 1 & N \\ ... & ... & ... & ... & N-1 & 1 \end{array}.$$

We first show that at this preference profile (due to ex-post efficiency) mechanism φ assigns objects with indices higher than 3 to the corresponding agents with certainty: for each $i > 3$ $\varphi_{a_i h_i}(\succ'') = 1$. Assume the opposite, namely that $\varphi_{a_j h_j} < 1$ for some $j > 3$. For φ to be ExPE there must be an efficient matching μ for which $\mu(h_j) = a_k \neq a_j$. We now show the inefficiency of any such matching by constructing another matching that dominates μ. Let $ind()$ denote index function such that for each $l \leq N$ $ind(a_l) = ind(h_l) = l$. Consider the chain C of agents coupled with corresponding houses that begins with (a_j, h_j) where the next agent in the chain is the agent assigned the house of the previous couple at μ : $C = (a_j, h_j), (a_k, h_k), (\mu(h_k), h_{ind(\mu(h_k))}), ...$ If at some point in C we face one of the first three agents, then the next agent in the chain by construction must be some agent a_m with the index above 3 that is assigned one of the first three houses (there is at least one house among the first three which is assigned to an "outsider" with an index higher than 3), $a_m : (m > 3) \cap (ind(\mu(a_m)) \leq 3)$.[16] Since N is finite and since each agent or object can appear only once in a matching, such a chain C inevitably arrives at the couple $(a_{ind(\mu(a_j))}, \mu(a_j))$ and constitutes a cycle that includes both a_j and h_j. Notice that all agents in C prefer the coupled houses to the houses assigned by μ. Therefore if they swap these houses according to C they arrive at a matching that dominates μ for all agents in C which contradicts the assumption that μ is efficient and that φ is ex-post efficient.

Finally it is left to be seen that for the preference profile $\succ''$ we can use the same arguments as for the case with only three agents as considered

[16]In other words we treat the first three agents and the first three houses as just one block-agent and one block-house as compared to others in order to avoid any exchanges between them. For instance, if at μ agent a_3 owns h_k, then there is some agent a_m that owns one of (a_1, a_2, a_3). After the transformation a_3 gives h_k away in exchange for this object previously owned by a_m.

above to show that ExPE, USP, and UEF are mutually incompatible. □

All three assumptions in the lemma are necessary. If we drop the ex-post efficiency requirement, a uniform lottery mechanism satisfies strategy-proofness and envy-freeness (and, therefore, upper shuffle-proofness and upper envy-freeness). If we drop the strategy-proofness requirement, then the probabilistic serial mechanism satisfies ex-post efficiency and envy-freeness (and upper envy-freeness). Finally, RSD is a natural benchmark to discuss the fairness requirement. It is easy to show that RSD is always SETE because of the underlying dictatorship procedure: the assignment probabilities for every house depend only on the preferences for the corresponding upper contour set.[17] At the same time RSD is not upper envy-freeness, which is true, for instance, for the preference profile $\succ$ in the proof above. The lemma shows that this gap between the strong equal treatment of equals and upper envy-freeness is so large that even the certain compromise on strategy-proofness (requiring upper shuffle-proofness instead of strategy-proofness) is not enough to close it.

Lemma 3.1 can be seen as a generalization of the statement in BM (p. 310) about the incompatibility of ex-post efficiency, strategy-proofness, and no envy for the case of three agents. Here we show the incompatibility of ex-post efficiency and two weaker properties: upper strategy-proofness and upper envy-freeness for any number of agents.[18]

In the following section we interchange the fairness and efficiency requirements: we relax the fairness criterion and strengthen the efficiency criterion in order to obtain a different but closely related impossibility result.

[17] This property is defined as a *weak invariance* in Hashimoto et. al (2014) and plays a central role in their characterization of the probabilistic serial mechanism.

[18] Perhaps BM did not show this impossibility result for the general case since they had a different focus: "For problems involving four agents and more, the impossibility result is more severe" (p. 310). However, the result they show (the incompatibility of strategy-proofness, ordinal efficiency and equal treatment of equals) is logically independent from Theorem 3.1 and especially from Lemma 3.1 since ordinal efficiency is stricter than ex-post efficiency.

3.4 Two Characterizations

We begin by characterizing the RSD mechanism as a unique strategy-proof, ex-post efficient, and weak envy-free mechanism for a problem with three agents.

Before we proceed, it is important to briefly mention the proof technique that is used in the proofs below. This technique usually involves *relabeling* the agents and objects in order to show the equivalence between different preference profiles. In general, we are not free to relabel the agents or the houses without changing the random assignment, as that would require the mechanism to be *neutral* toward the "name tags" of the agents and the houses so that the assignment is defined solely by the preference profile. We do not assume this type of neutrality. But if we use the properties of a mechanism (e.g., efficiency, strategy-proofness, fairness) in order to pin down specific values of some assignment probability, then since these properties should also hold for the same (or close enough) preference profile regardless of the name tags, we can find the same probability value for this other preference profile. In other words, all the mechanism's properties that we consider are essentially neutral, i.e., invariant with respect to any relabeling transformation. For instance, an ex-post efficient mechanism remains ex-post efficient regardless of any relabeling, a strategy-proof remains strategy-proof, and so forth. The following expresses this idea more formally:

Claim. If for some mechanism φ and some preference profile $\succ \in \mathcal{R}^N$ one can determine the value of some element in $\varphi_{ah}(\succ), a \in A, h \in H$ using the properties of φ, then this value $\varphi_{ah}(\succ)$ remains the same after any relabeling of agents and houses.

We now use the Claim in order to restrict our attention to only six types of preference profiles (since all other preference profiles are equivalent to one of these) and pin down all the random assignment probabilities.

Proposition 3.1. *(First characterization of RSD) For $N = 3$ a mechanism is strategy-proof, ex-post efficient, and weak envy-free among equals if and only if it is RSD.*

Proof. We know that RSD is strategy-proof, ex-post efficient, and satisfies weak envy-freeness among equals. We prove the other part of the characterization by sequentially checking all the preference profiles. Let φ be a strategy-proof, ExPE mechanism that also satisfies weak envy-freeness among equals.

For $N = 3$ there are the following six types of preference profiles (any other preference profile can be represented as one of these after the relabeling of agents and houses as discussed in the Claim above):

$$\text{type 1 (2 profiles):} \begin{cases} h_1 \succ_{a_1} h_3 \succ_{a_1} h_2 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 4 (1 profile):} \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_2 \succ_{a_3} h_3 \end{cases},$$

$$\text{type 2 (2 profiles):} \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_3 \succ_{a_2} h_2 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases}, \quad \text{type 5 (2 profiles):} \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_2 \succ_{a_3} (h_1, h_3) \end{cases},$$

$$\text{type 3 (1 profile):} \begin{cases} h_1 \succ_{a_1} h_2 \succ_{a_1} h_3 \\ h_1 \succ_{a_2} h_2 \succ_{a_2} h_3 \\ h_1 \succ_{a_3} h_3 \succ_{a_3} h_2 \end{cases}, \quad \text{type 6 (8 profiles):} \begin{cases} h_1 \succ_{a_1} (h_2, h_3) \\ h_2 \succ_{a_2} (h_1, h_3) \\ h_3 \succ_{a_3} (h_2, h_3) \end{cases}.$$

We begin with the profile of type 1. Since φ is ExPE we get $\varphi_{a_3 h_2} = 1$. Therefore agents a_1 and a_2 receive equal expected shares of the remaining houses $\varphi_{a_1 h_1} = \varphi_{a_2 h_1} = \varphi_{a_1 h_2} = \varphi_{a_2 h_2} = \frac{1}{2}$, otherwise one of them weakly envies another which contradicts the weak envy-freeness among equals.

In type 2, due to strategy-proofness agent a_2 receives the same expected share of house h_1 as before in type 1: $\varphi_{a_2 h_1} = \frac{1}{2}$. Using ExPE we get $\varphi_{a_2 h_2} = \varphi_{a_3 h_1} = 0$ and thus $\varphi_{a_1 h_1} = \varphi_{a_2 h_3} = \frac{1}{2}$. Suppose also $\varphi_{a_1 h_3} = x \in [0, \frac{1}{2}]$. Then the remaining probabilities are as follows: $\varphi_{a_1 h_2} = \varphi_{a_3 h_3} = \frac{1}{2} - x$ and $\varphi_{a_3 h_2} = \frac{1}{2} + x$.

Next, consider the preference profile of type 3. Since both agents a_1 and a_2 can transform this profile to one of type 2 considered above by switching their top objects, due to SP we get: $\varphi_{a_1 h_3} = \varphi_{a_2 h_3} = \frac{1}{2} - x$. (Here we implicitly used the Claim above.) Using weak envy-freeness among equals for these two agents and the fact that $\varphi_{a_3 h_2} = 0$ due to ExPE, we get $\varphi_{a_1 h_2} = \varphi_{a_2 h_2} = \frac{1}{2}$ and $\varphi_{a_1 h_1} = \varphi_{a_2 h_1} = x$. Consequently, the remaining expected share of house

h_1 goes to agent a_3: $\varphi_{a_3h_1} = 1 - 2x$.

Finally, consider the symmetric preference profile of type 4. Each agent can swap her second and third choices and transform the preference profile to that of type 3. Due to SP their expected shares of the top house h_1 are all equal: $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_3h_1} = 1 - 2x$. Therefore $x = \frac{1}{3}$ and the random assignments of types 1–4 are identical to RSD assignments.

Now that we have determined the unknown x it is easy to show that the random assignments for the remaining profiles are also equal to RSD. □

We get two immediate corollaries from the proposition by relaxing the weak envy-freeness among equals requirement.

Corollary 3.1. *(BM) For $N = 3$, a mechanism is strategy-proof, ex-post efficient, and satisfies the equal treatment of equals if and only if it is RSD.*

The second corollary follows from the fact that RSD satisfies weak envy-freeness (shown in BM):

Corollary 3.2. *For $N = 3$, a mechanism is strategy-proof, ex-post efficient and weak envy-free if and only if it is RSD.*

Note that upper shuffle-proofness would not have been enough for the proof when moving from the type 1 profile to the type 2 and also from the type 3 to the type 4. In fact, there we use *weak invariance* (Hashimoto et al., 2014) − a "part" of strategy-proofness complementary to upper shuffle-proofness, that requires the assignment probabilities to be fixed regardless of any changes in the lower contour set. Therefore, for $N = 3$ RSD can also be characterized as an ex-post efficient, weak envy-free among equals, upper shuffle-proof, and a weakly invariant mechanism.

We now complement the previous proposition by another characterization in which we use a slightly different fairness criterion: symmetry. Symmetry is defined using cardinal terms. Assume that agents report their utilities over objects and not just their ordinal preferences. Then a random assignment is symmetric if any two agents with identical utilities receive the assignments of the same expected utility.

Although symmetry is related to weak envy-freeness (equal agents receive individual assignments such that they do not dominate one another), these two properties are logically independent: symmetry applies to a smaller subset of utility domain, but for this set of utilities it also has stricter implications.

Proposition 3.2. *(Second characterization of RSD) For $N = 3$, a mechanism is strategy-proof, ex-post efficient, and symmetric if and only if it is RSD.*

The complete proof can be found in the appendix; it follows the same structure as the proof of the previous characterization. The main difference is that whenever we need to use weak envy-freeness among equals, it is sufficient to use the combination of symmetry and strategy-proofness instead.

One of the consequences of the proposition is that for the case of three agents RSD can also be characterized using anonymity, a property used by Gale (1987) in his conjecture, and also using a stronger equal treatment of equals (Corollary 3.1). Since RSD has this property and since anonymity implies symmetry, any mechanism that is strategy-proof, ex-post efficient, and anonymous is equivalent to RSD.

Another immediate consequence of the characterization is the impossibility (for the case of three agents) to find a mechanism that would Pareto dominate RSD (provide weakly higher expected utilities for all agents and strictly higher for at least one agent) and at the same time be symmetric and strategy-proof.

Corollary 3.3. *For $N = 3$, if a mechanism is strategy-proof and symmetric, it cannot dominate RSD.*

Given this result we can show the famous impossibility result by Zhou (1990): ex-ante efficiency, strategy-proofness, and symmetry are mutually incompatible.

Corollary 3.4. *(Zhou 1990) For $N \geq 3$, there does not exist a mechanism that is strategy-proof, ex-ante efficient and symmetric.*

For the case $N = 3$ the proof is just the combination of Corollary 3.3 and the fact that RSD is not ex-ante efficient but only ex-post efficient. For the general case $N \geq 3$, as it is done in the second part of the proof of the Theorem 3.1, we construct a preference profile such that any ex-post efficient mechanism cannot be cardinally dominated for any other than the first three agents. For this preference profile the problem is effectively reduced to the size of three.

In the next section we use Corollary 3.2 for the second impossibility result.

3.5 Second Impossibility Result

In the previous two sections we mostly discussed the problems with only three agents. For these cases the ex-post efficiency mechanism cannot be ordinally dominated, hence, ex-post efficiency coincides with ordinal efficiency. This changes when the number of agents is four or higher: an ex-post efficient mechanism such as RSD can be first-order stochastically dominated for some preference profiles. In the following two sections we further study the tradeoff between fairness and efficiency, where we put a higher weight on the latter and require ordinal efficiency and not just ex-post efficiency.

The next result shows the loss in fairness required to satisfy ordinal efficiency: any ordinally efficient mechanism must be either non-strategy-proof or cannot eliminate strong envy.

Theorem 3.2. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and weak envy-free.*

Proof. We prove by contradiction: assume that there exists a mechanism φ that is OE, SP, and wEF.

First note that it is enough to prove the claim for the problem where $N = 4$. For the case of more agents, consider the preference profiles similar to the type used in the proof of Theorem 3.1, namely, where the first four agents prefer the first four houses over all other houses, and other agents prefer the corresponding house of their own index to any other house. Due to ordinal efficiency all agents with indices higher than 4 receive the corresponding houses with certainty and the assignment problem is reduced to the size of four.

We begin with the following preference profile:

$$r(\succ^1) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 3^{\frac{1}{3}} & 4^0 \\ 3 & 2 & 4^0 & 1^1 \end{array}.$$

For example, the rank table for agent a_3 implies that she prefers house h_2 to all others (h_2 has rank 1) and receives 2/3 of this house in expectation, she prefers house h_1 to all others besides h_2 (i.e., h_1 has rank 2) and receives zero assignment probability of h_1, and so forth.

Due to the ordinal efficiency of φ and using Corollary 3.2 we find that $\varphi(\succ^1) = RSD(\succ^1)$. Indeed, agent a_4 is assigned house h_4 with certainty and we can repeat the same arguments used in the proof of Proposition 3.1 to determine the random assignment $\varphi(\succ^1)$.

Consider now two different preference profiles $\succ^2$ and $\succ'^2$:

$$r(\succ^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 2 & 1 & 4^0 & 3 \end{array}, r(\succ'^2) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 4^0 & 3 \end{array}.$$

Since φ is OE at $\succ^2$ and $\succ'^2$, at least two two – but not necessarily all four agents – receive zero probability of their worst houses (it is exactly for this reason that we need to consider two profiles and not just one). W.l.o.g. assume that these are agents a_3, a_4 for $\succ^2$ and a_2, a_4 for $\succ'^2$ (otherwise we can relabel the houses): $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$ and $\varphi_{a_2h_3}(\succ'^2) = \varphi_{a_4h_3}(\succ'^2) = 0$. We proceed with $\succ^2$ and for the profile $\succ'^2$ the argumentation line would be identical.

$$r(\succ^3) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2^0 & 1^{\frac{2}{3}} & 4^{\frac{1}{3}} & 3^0 \\ 3^0 & 2^0 & 4^0 & 1^1 \end{array}, r(\succ^4) = \begin{array}{cccc} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Now consider a preference profile $\succ^3$ that can be obtained from $\succ^2$ by changing the preferences of agent a_4 or from $\succ^1$ by changing the preferences of agent a_3.

On the one hand in the random assignment of agent a_4 $\varphi_{a_4h_1}(\succ^3) = \varphi_{a_4h_2}(\succ^3) = 0$ due to ExPE of φ and $\varphi_{a_4h_3}(\succ^3) = 0$ due to SP (otherwise agent a_4 might deviate to preference profile $\succ^2$). Therefore $\varphi_{a_4h_4}(\succ^3) = 1$ and $\varphi_{a_3h_4}(\succ^3) = 0$. On the other hand in the random assignment of agent a_3 due to SP $\varphi_{a_3h_1}(\succ^3) = 0$ and $\varphi_{a_3h_2}(\succ^3) = \frac{2}{3}$ as it was in the preference profile $\succ^1$.

Next consider the preference profile $\succ^4$ obtained from $\succ^3$ but where agents a_3 and a_4 have identical preferences.

Notice first that $\varphi_{a_3h_3}(\succ^4) = \frac{1}{3}$ remains the same as in $\succ^3$ due to SP. Secondly, due to ExPE $\varphi_{a_1h_4}(\succ^4) = \varphi_{a_2h_4}(\succ^4) = 0$ and $\varphi_{a_3h_1}(\succ^4) = \varphi_{a_4h_1}(\succ^4) = 0$. Thirdly, agent a_4 has to have the same random assignment as agent a_3 since their preferences are identical and we could therefore follow the same procedure where a_3 and a_4 are swapped (namely pick a_3 in $\succ^2$ and construct a profile analogous to $\succ^1$). Therefore $\varphi_{a_3h_4}(\succ^4) = \varphi_{a_4h_4}(\succ^4) = \frac{1}{2}$ and $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we will change the preferences of agents a_3 and a_4 sequentially so that they look symmetric to the preferences of a_1 and a_2. Consider the preference profile $\succ^5$ in which agent a_4 swaps her third and fourth best houses as compared to $\succ^4$.

$$r(\succ^5) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 3^0 & 2^{\frac{1}{6}} & 4^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}, \; r(\succ^6) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}, \; r(\succ'^6) = \begin{matrix} 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 1 & 2 & 3 & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{matrix}.$$

Note that $\varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and $\varphi_{a_4h_2}(\succ^5) = \frac{1}{6}$ due to SP and also that $\varphi_{a_1h_4}(\succ^5) = \varphi_{a_2h_4}(\succ^5) = 0$ and $\varphi_{a_3h_1}(\succ^5) = \varphi_{a_4h_1}(\succ^5) = 0$ due to ExPE. Therefore $\varphi_{a_3h_4}(\succ^5) = \varphi_{a_4h_4}(\succ^5) = \frac{1}{2}$ and using wEF for a_3 and a_4 we then get that $\varphi_{a_3h_2}(\succ^4) = \varphi_{a_4h_2}(\succ^4) = \frac{1}{6}$.

Now we do the same swap with houses h_1 and h_3 in the preferences of agent a_3 and get the preference profile $\succ^6$. We calculate her random assignment using the same argument as above.[19]

This result is derived from the fact that $\varphi_{a_3h_3}(\succ^2) = \varphi_{a_4h_3}(\succ^2) = 0$. But if we use the same procedure for $\succ'^2$ instead of $\succ^2$ then we get the random assignment for a profile $\succ'^6$.

The preference profile $\succ'^6$ is effectively identical to $\succ^6$ if we relabel houses h_1 and h_4 and agents a_1 and a_4. Due to the Claim at the beginning of this section we can conclude that agent a_2 at $\succ^6$ has to have the same random assignment as at $\succ'^6$: $\varphi_{a_2h_1}(\succ^6) = \varphi_{a_2h_4}(\succ'^6) = \frac{1}{2}$, $\varphi_{a_2h_2}(\succ^6) = \varphi_{a_2h_2}(\succ'^6) = \frac{1}{6}$ and $\varphi_{a_2h_3}(\succ^6) = \varphi_{a_2h_3}(\succ'^6) = \frac{1}{3}$. Then the full random assignment at $\succ^6$ is as follows:

$$r(\succ^6) = \begin{array}{cccc} 1^{\frac{1}{2}} & 2^{\frac{1}{2}} & 3^0 & 4^0 \\ 1^{\frac{1}{2}} & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 4^0 \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \\ 4^0 & 2^{\frac{1}{6}} & 3^{\frac{1}{3}} & 1^{\frac{1}{2}} \end{array}.$$

Finally, agent a_2 strongly envies agent a_1, which is a contradiction. □

It is easy to see the independence of axioms in Theorem 3.2. First, let us weaken the ordinal efficiency requirement and demand ex-post efficiency. Then there exists at least one ex-post efficient, strategy-proof, weak envy-free mechanism: the random serial dictatorship mechanism. Next, let us drop the weak envy-freeness requirement. Then there exists at least one strategy-proof, ordinally efficient mechanism: the serial dictatorship mechanism. Finally, the probabilistic serial mechanism is an example of an ordinally efficient, (weakly) envy-free mechanism.

This result is most strongly related to the impossibility result in BM, where instead of weak envy-freeness they use equal treatment of equals. Both these properties are natural relaxations of the (strict) envy-freeness, but, as I argue below, weak envy-freeness is a more practical fairness property than

[19] If in $\succ^6$ we relabel houses h_1 and h_4 and then swap agents a_1, a_2 and, on the other hand, a_3, a_4, then we get the same preference profile $\succ^6$. However, we would not be able to draw any conclusion regarding the random assignment for agents a_1 and a_2 at $\succ^6$ (agents a_3, a_4 after relabeling) since we did not determine the specific values and cannot use the logic of the Claim. For this reason we need a parallel procedure that begins with $\succ'^2$ and ends with $\succ'^6$.

equal treatment of equals (although not as handy to use in determining random assignments and therefore not as popular in the literature).

Weak envy-freeness is important for several reasons. First, if one agent does not weakly envy another, there always exist a set of Bernoulli utilities that are consistent with ordinal preferences for which envy-freeness is strict.[20] For instance, using the example in the Introduction, agent a_1 prefers her assignment to the other two whenever her preference for house h_1 is strong enough. Similarly, agent a_3 prefers her assignment whenever her dislike of house h_3 is strong enough.

Secondly, weak envy-freeness becomes even stronger in real-life applications, as compared to the case of abstract rational agents, if we account for bounded rationality. In the following I argue that once the agents receive their random assignments, their envy towards other agents decreases because they appreciate what they got. There is a vast related literature dealing with the so-called endowment effect, or the difference between the willingness to accept and the willingness to pay (WTA-WTP) for some goods. In these experiments, subjects value the goods that they are endowed with significantly more than the goods that they can purchase. This result holds in different settings and for different types of goods: lotteries over monetary outcomes, private goods such as coffee mugs or chocolate bars, and non-consumption goods such as decreased food risk and health insurance as well as public goods. There is, unfortunately, no WTA-WTP study for the case of *lotteries over non-consumption goods*, such as school slots, which would be the most relevant framework for the random assignment problem that we consider here. However, we can extrapolate the existing results. In their review of the WTA-WTP literature Horowitz and McConnell (2002) find that the

[20] This, however, cannot always be translated for the case of an entire random assignment since different pairwise comparisons might require mutually incompatible utilities. In the same example from the Introduction, for instance, if agent a_2 does not envy agent a_1, then she necessarily envies agent a_3. A random assignment for which such non-envy utilities exist is called *possibly envy-free*, which is stricter than weak envy-free. This distinction is not very common in the random assignment literature since most of the known weak envy-free mechanisms are also possible envy-free. Moreover, since our focus is on negative results, we also concentrate on the lighter notion of weak envy-freeness. For more detail on possible envy-freeness see Aziz et. al (2014).

average endowment effect for monetary lotteries is significant, but even more so for the non-consumption goods. For the case with monetary lotteries the WTA/WTP ratio is 2.10 (meaning that, on average, subjects are willing to sell a good for a price that is more than two times higher than the sum they are willing to pay for the same good), and the same ratio is 10.41 for the case of the non-consumption goods. This suggests the existence of a sizable WTA/WTP ratio for the lotteries over the non-consumption goods as well.[21] Therefore, one can expect that the agent is less likely to envy others due to the endowment effect because of the readjustment of her *cardinal* preferences ex-post – after being assigned a lottery. This readjustment can be relatively mild in the case of the weak envy-free assignment because it would not necessarily involve any change in *ordinal* preferences. For instance, in the example above, the agent a_1 after being endowed with her lottery can value house h_1 somewhat higher (or house h_3 – somewhat lower) so that she does not envy other agents. However, if a random assignment is *not* weak envy-free, then envy can be eliminated *only* if the agent also changes her ordinal preferences – since some other agent will have a stochastically dominant lottery.

Finally, the ex-ante judgment about fairness might often be crucial ex-post, even when the final assignment is completed. With a high probability, a non-weak-envy-free assignment induces a matching such that at least one agent is unhappy and can claim to be treated unfairly. For instance, if agent a_2 in the example above preferred house h_2 to all other houses, then this assignment is not weak-envy-free. If a_2 did not get her most preferred house h_2 ex-post (which happens with a 60% probability), she might justifiably claim to have been treated worse than agent a_3 since she got a stochastically dominated lottery. Once there is a legal basis for a lawsuit of some type of discrimination, it can be based exclusively on the verifiable information (reported preferences and the assigned lotteries) and not on the agent's private information (as in the case of envy-freeness). Clearly, it is important for the mechanism designer to avoid such risks.

[21] In a more recent study Isoni, Loomes, and Sugden (2011) find that the WTA/WTP gap is more robust for monetary lotteries than for coffee mugs even in settings specifically designed to neutralize any slight misconceptions of agents. See Fehr et al. (2015) for the most recent debate.

Overall, when the fairness of a random assignment is judged by comparing the individual assignments between each other, weak envy-freeness is arguably a reasonable minimum fairness requirement. In the following section, we discuss a different approach to fairness, where the individual assignments are compared to an alternative "fair" assignment of the equal division.

3.6 Third Impossibility Result

The last impossibility result also uses a strong notion of efficiency and a weak notion of fairness, but this time fairness is defined by equal division lower bound.

Theorem 3.3. *For $N \geq 4$ there does not exist a mechanism that is ordinally-efficient, strategy-proof, and satisfies equal division lower bound.*

Proof. The proof is by contradiction.[22] Assume that there exists a mechanism φ that satisfies ordinal efficiency, strategy-proofness, and equal division lower bound.

As before, we first prove the claim for the case $N = 4$, which can be generalized for a higher number of agents using certain preference profiles. These preference profiles have the following rank tables, we consider them sequentially:

$$r(\succ^1) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{matrix}, \quad r(\succ^2) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 3^0 & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{matrix}, \quad r(\succ^3) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{2}} & 4^0 & 3^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{1}{2}} & 4^{\frac{1}{4}} \end{matrix},$$

$$r(\succ^4) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4^0 & 3^{\frac{1}{2}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4^0 & 3^{\frac{1}{2}} \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{2}} & 4^0 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{2}} & 4^0 \end{matrix}, \quad r(\succ^5) = \begin{matrix} 1^{\frac{1}{4}} & 2 & 4^0 & 3 \\ 1^{\frac{1}{4}} & 2 & 4^0 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 3^{\frac{1}{4}} & 4^{\frac{1}{4}} \\ 1^{\frac{1}{4}} & 3^0 & 2^{\frac{3}{4}} & 4^0 \end{matrix}, \quad r(\succ^6) = \begin{matrix} 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 2^{\frac{1}{4}} & 4 & 3 \\ 1^{\frac{1}{4}} & 3^{\frac{1}{4}} & 2 & 4 \end{matrix}.$$

First, consider the preference profile $\succ^1$.

[22] This result can also be observed using the proof of Theorem 2 in the paper Athanassoglou and Sethuraman (2011). The focus in that paper is on a different type of object allocation problem – when agents have fractional endowments. I am grateful to Acelya Altuntas for pointing out this observation.

As before, the superscripts denote the random assignment $\varphi(\succ^1)$. Indeed, due to EDLB each agent has a right to receive at least $\frac{1}{4}$ of her most preferred house h_1 and at most $\frac{1}{4}$ of her least preferred house h_4. Then, due to ordinal efficiency, either $\varphi_{a_1h_3}(\succ^1) = \varphi_{a_2h_3}(\succ^1) = 0$ or $\varphi_{a_3h_2}(\succ^1) = \varphi_{a_4h_2}(\succ^1) = 0$ and, as it turns out, both conditions hold.

Consider now a profile $\succ^2$ that is derived from the previous profile using the swap of houses h_3 and h_4 in the preferences of agent a_1.

The random assignment $\varphi(\succ^2)$ is the same as before for the following reasons. First, the random assignment of house $h_1$is symmetric due to EDLB. Second, $\varphi_{a_1h_2}(\succ^2) = \frac{1}{2}$ because of SP (otherwise agent a_1 might deviate from/to $\succ^1$). Third, $\varphi_{a_1h_3}(\succ^2) = 0$ due to ExPE, implied by ordinal efficiency. As a result, we find the remaining element $\varphi_{a_1h_4}(\succ^2) = \frac{1}{4}$. Therefore, the random assignment of house h_4 is again symmetric due to EDLB. Finally, using the ordinal efficiency argument we find the random assignment of houses h_2 and h_3: $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ and $\varphi_{a_3h_2}(\succ^2) = \varphi_{a_4h_2}(\succ^2) = 0$ (again: only one of these conditions has to be satisfied due to OE, but in fact both of them hold because of the previous findings).

Next, consider the preference profile $\succ^3$ derived using the same swap of houses h_3 and h_4 but this time for agent a_2.

It turns out that the random assignment is again the same. First, $\varphi_{a_1h_3}(\succ^2) = \varphi_{a_2h_3}(\succ^2) = 0$ due to ExPE. Second, both $\varphi_{a_1h_2}(\succ^3)$ and $\varphi_{a_2h_2}(\succ^3)$ are equal to $\frac{1}{2}$ because of SP (otherwise one of the two agents a_1,a_2 would have switched from/to preference profile $\succ^2$). The rest of the random assignment can be found using EDLB as before.

Next, we consider a different preference profile $\succ^4$ in which the agents have opposite tastes regarding the other pair of houses: h_3 and h_4 (and not h_2 and h_3 as before).

The random assignment $\varphi(\succ^4)$ can be determined using the same argumentation line as in the case of $\succ^1$.

Finally, we consider the preference profile $\succ^5$, which can be derived from the profile $\succ^4$ using a swap of houses h_2, h_3 in the preferences of agent a_4, *and at the same time* from the profile $\succ^3$ using the swap of houses h_2, h_3 in the preferences of agent a_3.

The random assignment $\varphi(\succ^5)$ can be determined using the following arguments. First, since φ is SP, the elements $\varphi_{a_3h_4}(\succ^5)$ and $\varphi_{a_4h_4}(\succ^5)$ must correspond to the elements of $\varphi(\succ^3)$ and $\varphi(\succ^4)$ respectively: $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_4h_4}(\succ^5) = 0$. Second, we apply the ordinal efficiency argument to houses h_3 and h_4 and find that since $\varphi_{a_3h_4}(\succ^5) = \frac{1}{4} > 0$, the corresponding probabilities of agents a_1, a_2 are zero: $\varphi_{a_1h_3}(\succ^5) = \varphi_{a_2h_3}(\succ^5) = 0$. Third, due to ExPE $\varphi_{a_4h_2}(\succ^5) = 0$. Fourth, the assignment of house h_1 is identical due to EDLB. Therefore $\varphi_{a_4h_3}(\succ^5) = \frac{3}{4}$ and then $\varphi_{a_3h_3}(\succ^5) = \frac{1}{4}$ and $\varphi_{a_3h_2}(\succ^5) = \frac{1}{4}$.

So far there is no contradiction with our assumptions. However, the fact that $\varphi_{a_3h_2}(\succ^5)$ equals $\frac{1}{4}$ and is therefore different from $\varphi_{a_1h_2}(\succ^5)$ or $\varphi_{a_2h_2}(\succ^5)$ (since their sum has to be equal to one) contradicts the strategy-proofness of φ. Indeed, consider the profile $\succ^6$ which is different from $\succ^5$ in that agent a_3 swaps her preferences for houses h_3 and h_4 and thus becomes identical to agents a_1 and a_2.

Since any of the agents a_1, a_2, a_3 could swap their least preferred houses h_3, h_4 in order to deviate from/to $\succ^5$, due to strategy-proofness of φ we conclude that $\varphi_{a_1h_2}(\succ^6) = \varphi_{a_2h_2}(\succ^6) = \varphi_{a_3h_2}(\succ^6) = \frac{1}{4}$ and therefore $\varphi_{a_4h_2}(\succ^5) = \frac{1}{4}$ which contradicts the ex-post efficiency of φ. $\square$

We can easily check the independence of the axioms in this result. First, the pure lottery mechanism is strategy-proof and satisfies equal division lower bound, but it is not ordinally efficient. Second, the serial dictatorship mechanism is strategy-proof and ex-ante efficient (and therefore ordinally efficient), but does not satisfy equal division lower bound. Finally, the probabilistic serial mechanism is ordinally efficient and envy-free (and therefore satisfies equal division lower bound), but is not strategy-proof.

As in the case with weak envy-freeness, one can also argue that equal division lower bound is a relevant fairness concept. From the practical point of view, equal division lower bound appears to be important for two main reasons. First, equal division seems to be the most natural fair assignment and thus a natural benchmark to compare all other random assignments to. Secondly, equal division is often used in practice – whenever the assignment is made in the absence of or regardless of the data on agents' preferences, for instance. This is the case in the process of assigning Japanese teachers to Japanese schools abroad (Nihonjin gakkō). Each successful applicant is sent for two to three years to one of more than 80 schools all over the world regardless of his or her actual preferences.

From the theoretical point of view, equal division lower bound is related more to how efficient rather that how equitable the assignment is, as compared to weak envy-freeness and equal treatment of equals. Unlike the other two notions, EDLB does not compare the individual assignments to each other but to the (usually inefficient) equal division benchmark. Therefore, EDLB does not require the assignment to be fair in the egalitarian sense, but only that this assignment *dominates* the most egalitarian assignment – equal division.

Another essential feature of equal division lower bound is that several popular mechanisms satisfy this property. One of these mechanisms is RSD. Indeed, in the RSD procedure each agent has an equal chance of being the first in the ordering (and thus receiving her first best house), the second (and thus receiving at least her second best) and so on. Therefore, under the RSD assignments all agents are weakly better off than under the uniform lottery. Hence, an important implication of Theorem 3.3 is the restriction that it puts on the feasibility set of mechanisms that dominate RSD.

Corollary 3.5. *For $N > 3$ any ordinally efficient mechanism that dominates RSD is not strategy-proof.*

The corollary, however, does not restrict the set of mechanisms that dominate RSD *without* being ordinally efficient. Thus, in the set of strategy-proof mechanisms there might still be room for improvement upon RSD.

3.7 Conclusions

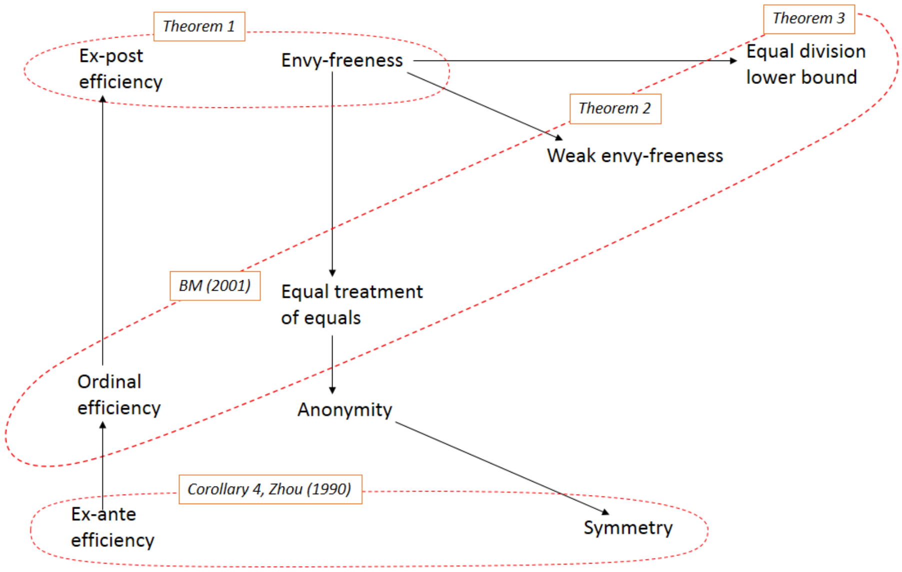
This paper considers the standard random assignment problem of assigning N indivisible objects to N agents and shows the impossibility for a strategy-proof mechanism to be simultaneously fair and efficient (in three specific ways). Theorem 3.1 shows the impossibility of combining a weak notion of efficiency – ex-post efficiency, with a strong notion of fairness – envy-freeness; it is the first known impossibility result in the related literature that involves ex-post efficiency. Theorem 3.2 shows the impossibility for the opposite set of properties: a weak notion of fairness – weak envy-freeness and a strong notion of efficiency – ordinal efficiency. Finally, Theorem 3.3 shows a similar impossibility result with a different weak fairness notion: equal division lower bound.

The paper also shows that for the case of three agents the trinity of strategy-proofness, ex-post efficiency, and weak envy-freeness for agents with identical preferences uniquely defines the random serial dictatorship mechanism. Alternatively, if we use symmetry—a cardinal fairness notion—instead of the ordinal weak envy-freeness among equals, we get the same characterization of the random serial dictatorship mechanism.

The first theorem is, perhaps, of the highest importance for the practical implementation of matching and random assignment mechanisms since it deals with the commonly required properties of strategy-proofness and ex-post efficiency. The other two theorems resemble the impossibility result of Bogomolnaia and Moulin (2001), although, perhaps, with two more relevant notions of fairness.

The results of the paper also fit in the recent literature that supports the central role of RSD among other mechanisms. This literature shows the equivalence of RSD to versions of other mechanisms used in practice. For example, Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to the *core from random endowments* mechanism, that initially randomly allocates objects and then proceeds by using the *top trading cycles (TTC)* algorithm in which agents voluntarily exchange the objects that they are endowed with. A recent paper by Bade (2014) generalizes this equivalence result to the set of all symmetrized Pareto optimal, strategy-proof, and non-bossy mechanisms. Finally, in specific cases, the equivalence holds for the celebrated *deferred acceptance* mechanism introduced by Gale and Shapley (1962), which is often used for the *two-sided matching problems* such as the school choice problem (as well as the college admission problem and job placement problem). The mechanism is equivalent to RSD in case schools

Figure 3.7.1: Tradeoff between fairness and efficiency in a random assignment: Impossibilities.



Notes: Dashed lines denote the mutual incompatibility in the class of strategy-proof mechanisms. The dashed line in the middle corresponds to three impossibilities at once: of ordinal efficiency and one of the contained fairness notions; it also applies to the case $N \geq 4$ whereas two other results apply for $N \geq 3$.

are initially indifferent between students and the ties are broken randomly for all schools together.

This paper also supports the use of RSD in random assignment problems, when strategy-proofness is of high importance. As argued throughout the paper, strategy-proofness, ex-post efficiency and weak envy-freeness are strongly desirable properties for a mechanism used in real-life applications, while equal division lower bound might be important when switching from one assignment procedure to another. Not only does RSD possess all four of these properties, but also, as this paper demonstrates, it is impossible to improve on any of the weak properties without violating another: to demand ordinal efficiency instead of ex-post efficiency, or envy-freeness instead of weak envy-freeness.

The central role of RSD as the strategy-proof mechanism becomes even more apparent in the problem with three agents. As this paper demonstrates, RSD is the unique strategy-proof and ex-post efficient mechanism that satisfies some of the weakest (among those presented here) fairness notions: symmetry or weak envy-freeness among equals. It, however, remains unclear, what combination of properties characterizes RSD for the general case. The characterization in this paper cannot be directly generalized even for the case of four agents (however, there are also no counter examples found). The reason for this complication is that weak envy-freeness (and especially weak envy-freeness among equals) is not handy enough as compared to the equal treatment of equals. For instance, for two agents with identical preferences weak envy-freeness gives precise implications only in case these agents receive identical probabilities for all but two objects. Then the two agents have to have the same random assignment for the remaining objects as well. Equal treatment of equals, on the contrary, has implications for the assignment probabilities of all objects. Therefore, I believe, generalizing this characterization would be more difficult than the result that uses equal treatment of equals.

Another open question is to what extent one of the three properties can be satisfied should the other two be taken at their extreme. For instance, if ordinal efficiency and envy-freeness are satisfied, then the probabilistic serial mechanism appears to be the "most" strategy-proof mechanism since it is weakly invariant (limits the set of profitable deviations) and weakly strategy-proof (which means that no agent can receive a stochastically dominant assignment by manipulating). Similarly, one could be interested in the "most fair" mechanism that satisfies strategy-proofness and ordinal efficiency (since the only known SD mechanism is very unfair), and in the "most" efficient mechanism that satisfies strategy-proofness and envy-freeness (again, the only known equal division or pure lottery mechanism disregards preferences and is therefore almost always inefficient).

3.8 Appendix: Proofs

Proof of the Remark in section 2.

Proof. For each statement in the proof, one can easily design an example sufficient to prove the statement. We try to prove them in general or at least to give understanding as to what examples may be helpful.

Envy-freeness$\Longrightarrow$ upper envy-freeness. We need to show that for each envy-free random assignment P it follows that for each $a, a' \in A$ and each $h \in H$ if $U(\succ_a, h) = U(\succ_{a'}, h)$ then $P_{ah} = P_{a'h}$. First note that $F(\succ_a, h, P_a) = F(\succ_{a'}, h, P_{a'})$ since otherwise one of the two agents might envy another (e.g., if she is almost indifferent between all objects in her upper contour set of h). Then notice that $F(\succ_a, h_a, P_a) = F(\succ_{a'}, h_{a'}, P_{a'})$ where h_a and $h_{a'}$ are the least preferred objects in $U(\succ_a, h)\backslash\{h\}$ and $U(\succ_{a'}, h)\backslash\{h\}$ respectively − for the same reason as earlier. Finally $P_{ah} = F(\succ_a, h, P_a) - F(\succ_a, h_a, P_a)$ and $P_{a'h} = F(\succ_{a'}, h, P_{a'}) - F(\succ_{a'}, h_{a'}, P_{a'})$ which completes the proof.

Upper-envy-freeness $\Longrightarrow$ strong equal treatment of equals. Here we need to show that for each upper envy-free random assignment P it follows that for each $a, a' \in A$ with identical preferences down to some $h \in H$ the random assignment down to this h is the same or, more formally, for each $h' \in H$ such that $h' \succ_a h$ and $h' \succ_{a'} h$ it follows that$P_{ah'} = P_{a'h'}$. We prove by induction: consider the top object $h_1 : h_1 \succ_a h'$ for each $h' \in H$ (and $h_1 \succ_{a'} h'$ since the preferences down to h are identical). Using the upper envy-freeness for

h_1 (since $U(\succ_a, h_1) = U(\succ_{a'}, h_1)$) we get $P_{ah_1} = P_{a'h_1}$. We then do it for the second top object and so forth until we reach h which would complete the proof.

Strong equal treatment of equals $\Longrightarrow$ *equal treatment of equals.* For ETE we need to consider only agents with identical preferences. Clearly, for any two of these agents the strong equal treatment of equals implies equal treatment of equals since SETE applies to all objects.

Envy-freeness $\Longrightarrow$ *weak envy-freeness.* This is true since if agents prefer their own assignments, then none of them strictly prefers the assignment of someone else.

Envy-freeness $\Longrightarrow$ *equal division lower bound.* Consider some agent $a \in A$ and her top object $h_1 \in H$. Since the assignment P is envy-free there is no agent a' with $P_{a'h_1} > P_{ah_1}$ (otherwise a could possibly envy a'). Therefore agent a gets at least her fair share of object h_1 of $\frac{1}{N}$. Next, consider the two top objects $\{h_1, h_2\}$ of agent a. Similarly, there is no agent a' with the total probability $(P_{a'h_1} + P_{a'h_2})$ higher than the total probability of agent a for the same two objects (otherwise a would envy a' once she is indifferent between h_1 and h_2 and does not care as much about the rest). Therefore the total probability $(P_{ah_1} + P_{ah_2})$ is at least as high as the fair share $\frac{2}{N}$. We use the same logic for the other objects and find that agent a is weakly better off under P than under the equal division.

Equal division lower bound $\Longrightarrow$*anonymity.* Whenever anonymity applies to a subset of agents, equal treatment of equals applies as well and has the same consequences. The opposite is not true.

Anonymity $\Longrightarrow$*symmetry.* If two agents with identical utilities receive identical random assignments, their expected utilities also coincide. The opposite is not true.

Independence of properties. Finally, it is left to show the mutual independence of the weak notions of fairness which is fairly easy to do by a contour example for each two notions. Indeed, these examples are easy to come up with since all the notions have a different nature: UEF, SETE, ETE, anonymity and symmetry can be applied to those preference profiles in which for some agents the preferences are (partially) identical; these properties require the corresponding assignment probabilities to be equal. On the other hand, wEF and EDLB apply to all preference profiles and do not require equalities. Comparing wEF and EDLB, wEF compares assignments between different agents, while EDLB compares them to the fair division. □

Proof of the Proposition 3.2 (Second characterization of RSD).

Proof. The necessity part follows from the fact that RSD is strategy-proof, ex-post efficient and satisfies symmetry. We prove the sufficiency part by checking sequentially all the preference profiles. Let φ be SP, ExPE and weak envy-free among equals mechanism.

For $N = 3$ there are the same six types of preference profiles as in the proof of Proposition 3.1.

Type 1. Since φ is ExPE we get $\varphi_{a_3h_2} = 1$. If $u_1 = u_2$ then due to symmetry $\varphi_1(u) = \varphi_2(u)$, otherwise the agents cannot possibly get equal utilities. If, on the contrary, $u_1 \neq u_2$ then one of the agents mimics the other to get a more preferred equal split of objects h_1 and h_3 unless $\varphi_1(u) = \varphi_2(u)$ already. Due to strategy-proofness of φ this mimic deviation is forbidden and therefore $\varphi_1(u) = \varphi_2(u) = RSD_1(u) = RSD_2(u)$, where the latter denotes the random assignment induced by RSD.

Type 2. Due to the previous result and due to strategy-proofness, $\varphi_{a_2h_1} = \frac{1}{2}$. Using ExPE we get $\varphi_{a_2h_2} = \varphi_{a_3h_1} = 0$ and thus $\varphi_{a_1h_1} = \varphi_{a_2h_3} = \frac{1}{2}$. Suppose also $\varphi_{a_1h_3} = x \in [0, \frac{1}{2}]$. Then the remaining probabilities are as follows: $\varphi_{a_1h_2} = \varphi_{a_3h_3} = \frac{1}{2} - x$ and $\varphi_{a_3h_2} = \frac{1}{2} + x$. Note that x cannot effectively depend on the reported utilities u_1, u_3 (given that the ordinal preferences are the same), otherwise φ is not strategy-proof.

Type 3. Due to the results for type 2 and due to strategy-proofness, $\varphi_{a_1h_3} = \varphi_{a_2h_3} = \frac{1}{2} - x$. If $u_1 = u_2$ then due to symmetry $\varphi_1(u) = \varphi_2(u)$, otherwise the agents cannot possibly get equal utilities. If $u_1 \neq u_2$ then due to strategy-proofness $\varphi_1(u) = \varphi_2(u)$, otherwise one of the two agents can profitably mimic another. Since $\varphi_{a_3h_2} = 0$ due to ExPE, we get $\varphi_{a_1h_2} = \varphi_{a_2h_2} = \frac{1}{2}$ and $\varphi_{a_1h_1} = \varphi_{a_2h_1} = x$. Consequently, the remaining expected share of house h_1 goes to agent a_3: $\varphi_{a_3h_1} = 1 - 2x$.

Type 4. Finally, consider the symmetric preference profile of type 4. Each agent can swap her second and third choices and transform the preference profile to that of type 3. Due to SP their expected shares of the top house h_1 are all equal: $\varphi_{a_1h_1} = \varphi_{a_2h_1} = \varphi_{a_3h_1} = 1 - 2x$. Therefore $x = \frac{1}{3}$ and the random assignments of types 2 and 3 are identical to RSD assignments.

Continue with type 4. The top object h_1 is equally split, hence it is left to check how object $h_2$is assigned. Assume for a contradiction that for some u consistent with type 4 ordinal preferences h_2 is not equally split.

First let us check that this is not possible if two agents have identical utilities and the third differs: w.l.o.g. $u_1 = u_2 \neq u_3$ and due to symmetry and our assumption $\varphi_{a_1h_2}(u) = \varphi_{a_2h_2}(u) \neq \varphi_{a_3h_2}(u)$. If now agent a_3 mimics the other two agents, he gets an equal share of 1/3 due to the symmetry. But this share cannot differ from $\varphi_{a_3h_2}(u)$, otherwise a_3 could profitably switch in one of the directions. Therefore in this case all objects are equally split.

Next consider the case when all three utilities differ and, w.l.o.g. let agents a_1 and a_2 get different shares of h_2 such that $\varphi_{a_1h_2}(u) \neq 1/3$. Consider a different utility profile $u' : (u'_{-1} = u_{-1}) \cap (u'_1 = u_2)$ which is also consistent with type 4 ordinal preferences. In $\varphi(u')$ agents a_1 and a_2 get the same shares of h_2 $\varphi_{a_1h_2}(u') = \varphi_{a_2h_2}(u') = 1/3$ due to the previous observation. Therefore a_1 can profitably switch between u and u' in one of the directions, which is in contradiction with strategy-proofness.

Type 5. Due to strategy-proofness agents a_1 and a_2 get the same shares of h_1 as in type 2 profile, and agent a_3 gets none of h_1 due to ExPE. As before, due to symmetry and strategy-proofness, agents a_1 and a_2 get equal shares of the other two objects as well. The other probabilities corresponding to type 5 profile can be easily determined, they also coincide with those of RSD.

Type 6. Here φ coincides with RSD due to ExPE. $\square$

Bibliography

1. Abdulkadiroğlu, A. (2013). School choice. The Handbook of Market Design.

2. Abdulkadiroğlu, A., Che, Y. K., & Yasuda, Y. (2011). Resolving Conflicting Preferences in School Choice: The "Boston Mechanism" Reconsidered. The American Economic Review, 399-410.

3. Abdulkadiroğlu, A., Che, Y. K., & Yasuda, Y. (2015). Expanding "choice" in school choice. American Economic Journal: Microeconomics, 7(1), 1-42.

4. Abdulkadiroğlu, A., & Sönmez, T. (1998). Random serial dictatorship and the core from random endowments in house allocation problems. Econometrica, 689-701.

5. Abdulkadiroğlu, A., & Sönmez, T. (2003). Ordinal efficiency and dominated sets of assignments. Journal of Economic Theory, 112(1), 157-172.

6. Abdulkadiroğlu, A., & Sönmez, T. (2003). School choice: A mechanism design approach. The American Economic Review, 93(3), 729-747.

7. Abdulkadiroğlu, A., & Sönmez, T. (2010). Matching markets: Theory and practice.

8. Arnsperger, C. (1994). Envy-freeness and distributive justice. Journal of Economic Surveys, 8(2), 155-186.

9. Athanassoglou, S., & Sethuraman, J. (2011). House allocation with fractional endowments. International Journal of Game Theory, 40(3), 481-513.

10. Aziz, H., Gaspers, S., Mackenzie, S., & Walsh, T. (2014, May). Fair assignment of indivisible objects under ordinal preferences. In Proceedings of the 2014 international conference on Autonomous agents and multi-agent systems (pp. 1305-1312). International Foundation for Autonomous Agents and Multiagent Systems.

11. Bade, Sophie. "Random Serial Dictatorship: The One and Only." Royal Holloway College, Unpublished mimeo (2014).

12. Birkhoff, G. (1946). Three observations on linear algebra. Univ. Nac. Tucumán. Revista A, 5, 147-151.

13. Bogomolnaia, A., & Moulin, H. (2001). A new solution to the random assignment problem. Journal of Economic Theory, 100(2), 295-328.

14. Budish, E., Che, Y. K., Kojima, F., & Milgrom, P. (2013). Designing random allocation mechanisms: Theory and applications. The American Economic Review, 103(2), 585-623.

15. Che, Y. K., & Kojima, F. (2010). Asymptotic equivalence of probabilistic serial and random priority mechanisms. Econometrica, 78(5), 1625-1672.

16. Chen, Y., & Sönmez, T. (2002). Improving efficiency of on-campus housing: An experimental study. American Economic Review, 1669-1686.

17. Coles, P., Cawley, J., Levine, P. B., Niederle, M., Roth, A. E., & Siegfried, J. J. (2010). The job market for new economists: A market design perspective. The Journal of Economic Perspectives, 24(4), 187-206.

18. Dubins, L. E., & Freedman, D. A. (1981). Machiavelli and the Gale-Shapley algorithm. American mathematical monthly, 485-494.

19. Dur, U. (2013). The modified Boston mechanism. Mimeo.

20. Erdil, A. (2014). Strategy-proof stochastic assignment. Journal of Economic Theory, 151, 146-162.

21. Erdil, A., & Ergin, H. (2008). What's the matter with tie-breaking? Improving efficiency in school choice. The American Economic Review, 98(3), 669-689.

22. Ergin, H., & Sönmez, T. (2006). Games of school choice under the Boston mechanism. Journal of Public Economics, 90(1), 215-237.

23. Fehr, D., Hakimov, R., & Kübler, D. (2015). The willingness to pay–willingness to accept gap: A failed replication of Plott and Zeiler. European Economic Review, 78, 120-128.

24. Foley, D. K. (1967). Resource allocation and the public sector. YALE ECON ESSAYS, VOL 7, NO 1, PP 45-98, SPRING 1967. 7 FIG, 13 REF.

25. Gale, D. (1987). College course assignments and optimal lotteries. University of California at Berkeley.

26. Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. American mathematical monthly, 9-15.

27. Harless, P. (2015). Immediate acceptance in school choice: Comparing implementations. Mimeo.

28. Hashimoto, T., Hirata, D., Kesten, O., Kurino, M., & Ünver, M. U. (2014). Two axiomatic approaches to the probabilistic serial mechanism. Theoretical Economics, 9(1), 253-277.

29. He, Y. (2012). Gaming the Boston school choice mechanism in Beijing. Manuscript, Toulouse School of Economics.

30. He, Y., Miralles, A., Pycia, M., & Yan, J. (2015). A Pseudo-Market Approach to Allocation with Priorities. Mimeo.

31. Horowitz, J. K., & McConnell, K. E. (2002). A review of WTA/WTP studies. Journal of Environmental Economics and Management, 44(3), 426-447.

32. Hugh-Jones, D., Kurino, M., & Vanberg, C. (2014). An experimental study on the incentives of the probabilistic serial mechanism. Games and Economic Behavior, 87, 367-380.

33. Hylland, A., & Zeckhauser, R. (1979). The efficient allocation of individuals to positions. The Journal of Political Economy, 293-314.

34. Isoni, A., Loomes, G., & Sugden, R. (2011). The willingness to pay – willingness to accept gap, the "endowment effect," subject misconceptions, and experimental procedures for eliciting valuations: Comment. The American Economic Review, 101(2), 991-1011.

35. Kesten, O. (2009). Why do popular mechanisms lack efficiency in random environments?. Journal of Economic Theory, 144(5), 2209-2226.

36. Kesten, O., & Kurino, M. (2013). Do outside options matter in school choice? A new perspective on the efficiency vs. strategy-proofness trade-off.

37. Kesten, O., Kurino, M., & Nesterov, A. (2015). Efficient lottery design (No. SP II 2015-203). WZB Discussion Paper.

38. Kesten, O., & Ünver, M. U. (2015). A theory of school-choice lotteries. Theoretical Economics, 10(2), 543-595.

39. Kesten, O., & Yazici, A. (2012). The Pareto-dominant strategy-proof and fair rule for problems with indivisible goods. Economic Theory, 50(2), 463-488.

40. Knuth, D. (1996). An exact analysis of stable allocation. Journal of Algorithms, 20, 431-442.

41. Kojima, F. (2009). Random assignment of multiple indivisible objects. Mathematical Social Sciences, 57(1), 134-142.

42. Kojima, F., & Manea, M. (2010). Incentives in the probabilistic serial mechanism. Journal of Economic Theory, 145(1), 106-123.

43. Manea, M. (2008). Random serial dictatorship and ordinally efficient contracts. International Journal of Game Theory, 36(3-4), 489-496.

44. Manea, M. (2009). Asymptotic ordinal inefficiency of random serial dictatorship. Theoretical Economics, 4(2), 165-197.

45. Mennle, T., & Seuken, S. (2014). The Naïve versus the Adaptive Boston Mechanism. arXiv preprint arXiv:1406.3327.

46. Mennle, T., & Seuken, S. (2015). An axiomatic approach to characterizing and relaxing strategyproofness of one-sided matching mechanisms. Mimeo.

47. Miralles, A. (2009). School choice: The case for the Boston mechanism. In Auctions, Market Mechanisms and Their Applications (pp. 58-60). Springer Berlin Heidelberg.

48. Moulin, H. (2014). Cooperative Microeconomics: A Game-Theoretic Introduction.

49. Nesterov, A. S. (2014). Fairness and efficiency in a random assignment: Three impossibility results (No. SP II 2014-211). WZB Discussion Paper.

50. Nesterov, A. S. (2015). School Choice With Advice: The Adaptive Acceptance Rule.

51. Pathak, P. A. (2011). The mechanism design approach to student assignment. Annu. Rev. Econ., 3(1), 513-536.

52. Pathak, P. A., & Sönmez, T. (2008). Leveling the playing field: Sincere and sophisticated players in the Boston mechanism. The American Economic Review, 98(4), 1636-1652.

53. Roth, A. E. (1982). The economics of matching: Stability and incentives. Mathematics of operations research, 7(4), 617-628.

54. Roth, A. E. (1984). The evolution of the labor market for medical interns and residents: a case study in game theory. The Journal of Political Economy, 991-1016.

55. Roth, A. E., Sonmez, T., & Utku Unver, M. (2004) "Kidney Exchange," Quarterly Journal of Economics 119(2): 457-488

56. Roth, A. E., Sönmez, T., & Utku Ünver, M. (2005). Pairwise kidney exchange. Journal of Economic Theory, 125(2), 151-188.

57. Shapley, L., & Scarf, H. (1974). On cores and indivisibility. Journal of mathematical economics, 1(1), 23-37.

58. Thomson, W. (2007). Fair allocation rules. Handbook of social choice and welfare, 2, 393-506.

59. Troyan, P. (2012). Comparing school choice mechanisms by interim and ex-ante welfare. Games and Economic Behavior, 75(2), 936-947.

60. Von Neumann, J. (1953). A certain zero-sum two-person game equivalent to the optimal assignment problem. Contributions to the Theory of Games, 2, 5-12.

61. Yılmaz, Ö. (2010). The probabilistic serial mechanism with private endowments. Games and Economic Behavior, 69(2), 475-491.

62. Zhou, L. (1990). On a conjecture by Gale about one-sided matching problems. Journal of Economic Theory, 52(1), 123-135.